Displacement/deformation method **without** Sign Conventions.

With this method the construction is divided into members and joints. The relation between member end forces and member end displacements deliver equations with the joint displacements as unknowns to be solved. Member end forces are 'forces and moments', joint displacements are joint translations and joint rotations.

The examples/exercises are worked out without the computer
For a plane construction a joint can translate/displace horizontally and vertically, and can rotate. So each joint has three unknows to be solved, for a construction with N joints N*3 equations have to be solved which can be done e.g. with the elimnation method of GAUSS. N*3 equations!, therefore the computer is used, easy solving a lot of equations, page 94 with the concerning computer code.

This method can be applied for statically determinate and indeterminate constructions, making all kinds of constructions easy to be calculated.

In the following pages the displacement method will be explained step by step, all without sign conventions. The 'drawn assumptions' , forces, moments and angles, determine the derivation of the equations to be solved. This way, not disturbed by sign conventions I have been able to write a program with which the examples are checked, all results ok.
(Part with grids page 81, no program not checked .)

Dear students, I hope having made the displacement method understandeble by avoiding sign conventions.

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The Netherlands.

1. Coinciding axially loaded members.

1.1. The relation between member end forces and joint displacements.

Fig.1.
The drawn construction consists of two mebers 1 and 2. The member ends are connected with the joints 1, 2 and 3.

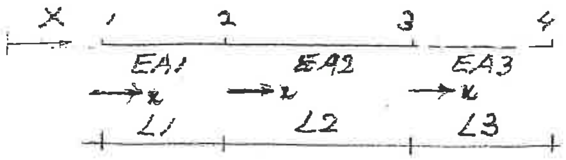


Fig.1.

E is the modulus of elasticity, kN/m^2,
A1 and A2 the cross section surfaces, m^2,
EA, E times A is the strain stiffness, EA1 for member 1 and EA2 for member 2, (kN/m^2)*(m^2).
The member lengyhs are L1 and L2.

The assumed direction of the member axes x is from lowest to highest member end number, here to the right.
The direction of the construction axis X is assumed to the right.

Fig.2.
On the member ends of the from the joints separated members act member end forces, F12 and F21, F23 and F32. Their assumed direction is like the member axis x of the members.

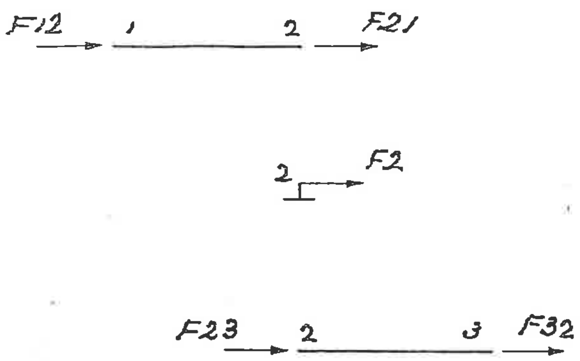


Fig.2.

Determining the member stiffness matrix S5.

Fig.3.
The joint displacements UA en UB, being the member end displacements as well, are assumed to the right like member axis x.

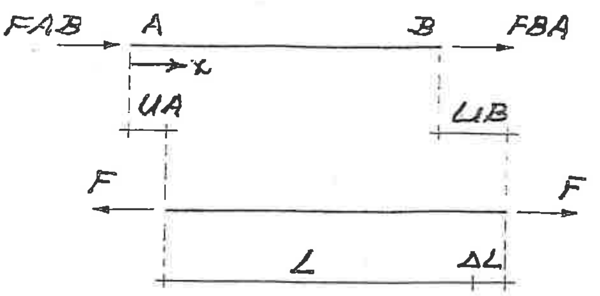


Fig.3.

There are two possibilities to derive the same relation between member end forces and joint displacements.

The first possibility.

Is UB larger than UA then the member becomes longer, $\Delta L=UB-UA$. The member is a tension member. At the member ends act tensile forces same size like the figure shows.

With Hooke's law follows $\Delta L=FL/EA$.
(F times L divided by E times A.)
From which follows $F=(EA/L)\Delta L$.
With member stiffness factor $R=EA/L$ is $F=R\Delta L$.

With $\Delta L=UB-UA$ is $F=R(UB-UA)$ or $F=R(-UA+UB)$.

Member end forces FAB and F of member end A are 'the same' forces, thus $F=-FAB$ or $FAB=-F$.

With $FAB=-F$ follows $FAB=-R(-UA+UB)$ or

$$FAB=R(UA-UB). \qquad 1)$$

Member end forces FBA and F of member end B are 'the same' forces, thus $F=FBA$ or $FBA=F$.

With $FBA=F$ follows $FBA=R(-UA+UB)$. 2)

These two equations show the relation between member end forces FAB and FBA, joint displacements UA and UB, by means of member stiffness factor $R=EA/L$.

$$\underset{\underline{f}}{\begin{bmatrix}FAB\\FBA\end{bmatrix}} = \underset{S5}{\begin{bmatrix}R & -R\\ -R & R\end{bmatrix}} \cdot \underset{\underline{u}}{\begin{bmatrix}UA\\UB\end{bmatrix}}$$

$$\begin{bmatrix}S5(1,1) & S5(1,2)\\ S5(2,1) & S5(2,2)\end{bmatrix}$$

Fig.4.

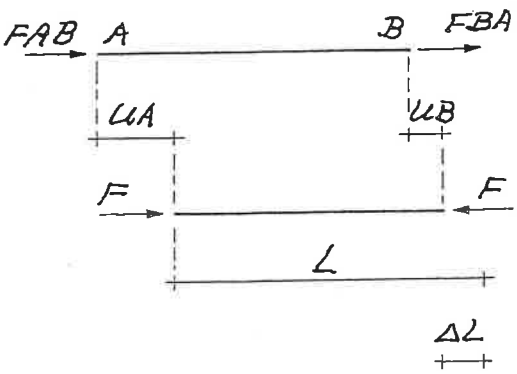
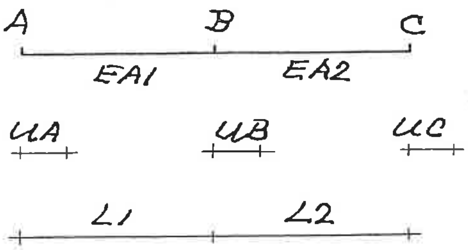


Fig.5.



R1=EA1/L1 R2=EA2/L2

$$\begin{bmatrix}FAB\\FBA\end{bmatrix} = \begin{bmatrix}R1 & -R1\\ -R1 & R1\end{bmatrix} \cdot \begin{bmatrix}UA\\UB\end{bmatrix}$$

$$\underset{\underline{f}}{\begin{bmatrix}FBC\\FCB\end{bmatrix}} = \underset{S5}{\begin{bmatrix}R2 & -R2\\ -R2 & R2\end{bmatrix}} \cdot \underset{\underline{u}}{\begin{bmatrix}UB\\UC\end{bmatrix}}$$

Fig.6.

Fig.4.
The two equations can be represented with
f = S5*u. In which is

f   the force vector (or force column),

S5   the member stiffness matrix, and

u   the displacement vector (column).

An element of f is equal to a row of S5 multiplied by column u.

FAB= S5(1,1)*UA + S5(1,2)*UB

R*UA   -R*UB

FBA= S5(2,1)*UA + S5(2,2)*UB

-R*UA   +R*UB

The member end forces FAB and FBA arise due to the joint displacements UA and UB.

The second possibility.
Fig.5.
Now displacement UA is larger than displacement UB instead of UB larger than UA.
The member is getting ΔL=UA-UB shorter, it is a compression member.
On the member ends act forces F with same size, the member is in equilibrium.

With Hooke's law follows ΔL=FL/EA.

from that follows   F=(EA/L)ΔL.

With member stiffness factor R=EA/L is F=RΔL.

With ΔL=UA-UB is   F=R(UA-UB).

Member end forces FAB and F of member end A, are the 'same forces', F=FAB or FAB=F.
With FAB=F follows   FAB=R(UA-UB)   1)

Member end forces FBA and F of member end B are 'the same' forces, thus F=-FAB or FBA=-F.

With FBA=-F follows   FBA=-R(UA-UB)   or

FBA=R(-UA+UB).   2)

The same equations are found (ofcourse) as for the tension member of figure 3.
The relation between member end forces and joint displacements is determined by strain stiffness EA and member length L, in other words, by member stiffness factor R=EA/L.
If the construction consists of one single member then construction stiffness matrix CC is equal to member stiffness matrix S5.

Fig.6.
If the construction consists of two members then one gets two times 2 equations as shown on the left in matrix form. Both systems of two equations can be composed into one system of three equations with three unknown displacements UA, UB and UC.

1.2. From member matrices S5 to construction matrix CC.

Fig.7.
Joints and members are separated from each other. The on the member ends acting member end forces are assumed to be directed to the right.

On the joints act the member end forces as large as but opposite directed, thus to the left. The member stiffness factors of member 1 and 2 are R1 and R2.

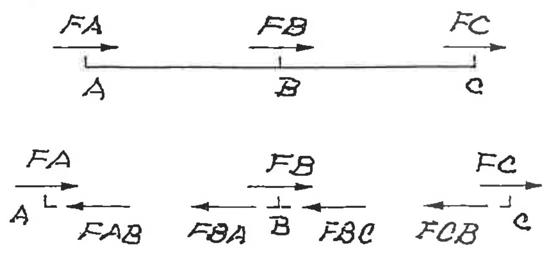


Fig.7.

On joint A acts, see fig.6,

FAB = R1*UA -R1*UB +0*UC 1)

On joint B acts,

FBA+FBC= -R1*UA +R1*UB +R2*UB -R2*UC
= -R1*UA + (R1+R2)*UB -R2*UC 2)

On joint C acts,

FCB = 0*UA -R2*UB +R2*UC 3)

This way arise three equations on the left shown in matrix form,

$$\underset{\underline{f}}{\begin{bmatrix} FAB \\ FBA+FBC \\ FCB \end{bmatrix}} = \underset{CC}{\begin{bmatrix} R1 & -R1 & 0 \\ -R1 & R1+R2 & -R2 \\ 0 & -R2 & R2 \end{bmatrix}} \cdot \underset{\underline{u}}{\begin{bmatrix} UA \\ UB \\ UC \end{bmatrix}}$$

with force vector $\underline{f}$,
construction stiffness matric CC, and
displacement vector $\underline{u}$.

Both systems of two equations can be written out as shown here below, and can be added.

FAB= R1*UA -R1*UB +0*UC 1')
FBA= -R1*UA +R1*UB +0*UC 2')
0 = 0*UA +0*UB +0*UC 3')

0 = 0*UA +0*UB +0*UC 1'')
FBC= 0*UA +R2*UB -R2*UC 2'')
FCB= 0*UA -R2*UB +R2*UC 3'')

Equation 1') and 1'') added gives equation 1), see above, etc.

Fig.8.
The joints are loaded with joint load forces FA, FB and FC, assumption directed to the right.
On the separated joints act also the to the left directed member end forces FAB, FBA, FBC and FCB.

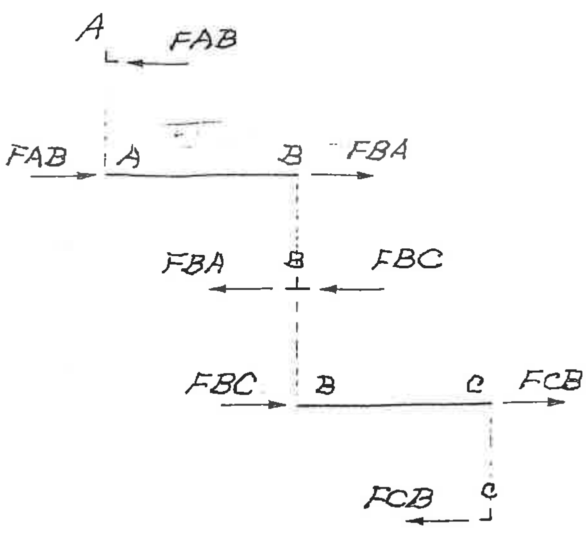


Fig.8.

Σ hor. joint A= 0

FA-FAB=0 or FAB=FA

Σ hor. joint B= 0

FB-FBA-FBC =0 or FBA+FBC=FB

Σ hor. joint C= 0

FC-FCB=0 or FCB=FC

On the left the three eqations are represented in matrix form.

$$\underset{CC}{\begin{bmatrix} R1 & -R1 & 0 \\ -R1 & R1+R2 & -R2 \\ 0 & -R2 & R2 \end{bmatrix}} \cdot \underset{\underline{u}}{\begin{bmatrix} UA \\ UB \\ UC \end{bmatrix}} = \underset{\underline{f}}{\begin{bmatrix} FA \\ FB \\ FC \end{bmatrix}}$$

Example.

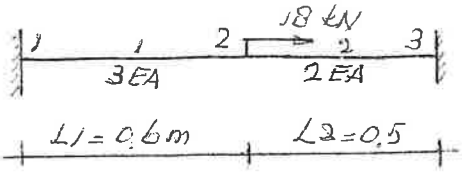


Fig 1.

Fig.1.
The statically indeterminate construction consists of 2 members and 3 joints. Yhe joints are numbered in arbitrary order. Member 1 with strain stiffness 3EA, member 2 with 2EA.
The member stiffness factors are

R1= EA1/L1= 3EA/0,6= 5EA and

R2= EA2/L2= 2EA/0,6= 4EA.

The joint load forces are
F1= 0 kN,   F2= 18 kN and F3= 0 kN.
The joint displacements are U1, U2 and U3, assumed direction to the right.

$$\begin{bmatrix} F12= \\ F21= \end{bmatrix} = EA \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \cdot \begin{bmatrix} U1 \\ U2 \end{bmatrix}$$

F12= EA(5*U1 -5*U2)

F21= EA(-5*U1 5*U2)

$$\begin{bmatrix} F23= \\ F32= \end{bmatrix} = EA \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \cdot \begin{bmatrix} U2 \\ U3 \end{bmatrix}$$

F23= EA(4*U2 -4*U3)

F32= EA(-4*U2 4*U3)

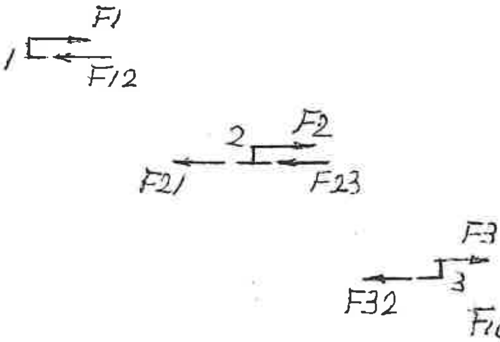


Fig. 2.

Fig.2.
On the left the two equations of the member end forces of both members are represented in matrix form. If the displacements U1, U2 and U3 are known then the member end forces F12 and F21, F23 and F32 can be calculated.
On the separated joints act the joint load forces, assumed to the right, F1=0 kN, F2= 18 kN and F3= 0 kN.
Like done on the preceding page is here given the relation between construction matrix CC, displacement vector <u>u</u> and force vector <u>f</u> in matrix form.

$$EA \underbrace{\begin{bmatrix} 5 & -5 & 0 \\ -5 & 5+4 & -4 \\ 0 & -4 & 4 \end{bmatrix}}_{CC} \cdot \underbrace{\begin{bmatrix} U1 \\ U2 \\ U3 \end{bmatrix}}_{\underline{u}} = \underbrace{\begin{bmatrix} 0 \\ 18 \\ 0 \end{bmatrix}}_{\underline{f}}$$

The equations written out are

EA(5*U1 -5*U2 +0*U3)= 0

EA(-5*U1 +9*U2 -4*U3)=18

EA(0*U1 -4*U2 +4*U3)= 0

When the three displacements are unknown the a solution is not possible, see figure 1. At least one displacement must be known. Here two displacements are known, U1=0 and U3=0.

The equations then become

EA(<u>1</u>*U1 -<u>0</u>*U2 +<u>0</u>*U3)= 0

EA(<u>0</u>*U1 +9*U2 -0*U3)=18

EA(<u>0</u>*U1 -<u>0</u>*U2 +<u>1</u>*U3)= 0

$$EA \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & -4 \\ 0 & 0 & 1 \end{bmatrix}}_{CC} \cdot \underbrace{\begin{bmatrix} U1 \\ U2 \\ U3 \end{bmatrix}}_{\underline{u}} = \underbrace{\begin{bmatrix} 0 \\ 18 \\ 0 \end{bmatrix}}_{\underline{f}}$$

In the concerning rows and columns of CC come zeros 0 and on the main diagonal ones 1 given in matrix form on the left. In the computer program construction matrix CC is changed this way while the size of CC does not change, the number of equations stays the same.
(The equations then are solved with e.g. the method of GAUSS.)
EA(0*U1 +9*U2 -0*U3)=18 or EA(9*U2)=18
from wich follows <u>U2=2/EA</u> in m.

Remark, an element of CC comes from member stiffness factor 'R=EA/L' with dimension (kN/m^2)(m^2)/m or kN/m, in the equation then (kN/m)*U2= kN and follows U2 in m. Thus <u>U2=2/EA</u> is a number in meters m.

$$\begin{bmatrix} F12= \\ F21= \end{bmatrix} = EA \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} U1 \\ U2 \end{bmatrix}$$

F12= EA(5*U1 -5*U2)

F21= EA(-5*U1 5*U2)

Fig.3a.

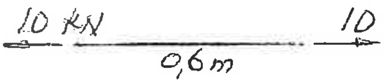


Fig.3b.

$$\begin{bmatrix} F23= \\ F32= \end{bmatrix} = EA \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \cdot \begin{bmatrix} U2 \\ U3 \end{bmatrix}$$

F23= EA(4*U2 -4*U3)

F32= EA(-5*U2 5*U3)

Fig.4a

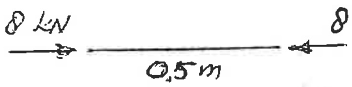


Fig.4b

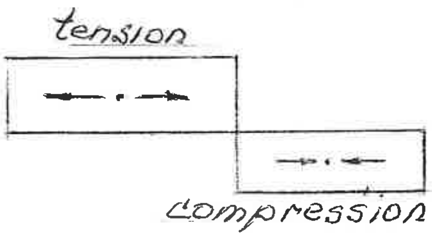


Fig5

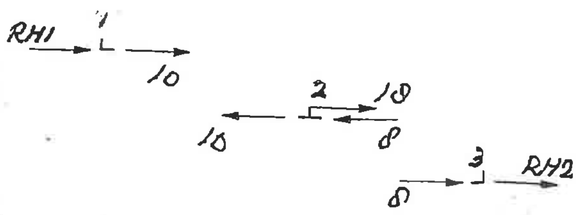


Fig.6.

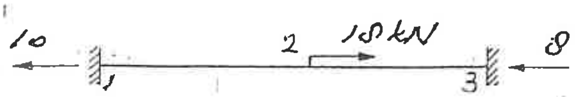


Fig.7.

Now that the displacements are known the member end forces can be calculated.

Fig.3a and 3b.
With the two equations for the first member follow with U1=0 and U2=a/EA

F12 =EA(5*U1 -5*U2)
    =EA(5*0 -5*2/EA)= EA(-10*EA)=-10 kN

The answer for F12 is negativ, so that the member end force is not directed to the right as assumed but to the left. The force does not press on member end 1 but pulls at end 1.

F21 =EA(-5*U1 +5*U2)
    =EA(-5*0 +5*2/EA)= EA(10/EA)= 10 kN

A positive answer, so that the member end force at member end 2 is as assumed directed to the right, pulls at member end 2.
Member 1 is a tension member. The elongation of the member is ΔL=10*0,6/3EA = 2/EA.

Fig.4a en 4b.
Similar way for member 2 with its member end displacements, being the joint displacements U2=2/EA and U3=0.

F23 =EA(4*U2 -4*U3)
    =EA(4*2/EA-4*0)= EA(8/EA)= 8 kN

A positiv answer, F23 is as assumed directed to the right. The force presses on member end 2.

F32 =EA(-4*U2+4*U3)
    =EA(-4*2/EA+4*0)=EA(-8/EA)= -8 kN

A negative answer so not as assumed to the right but to the left. The force pushes on member end 3. Member 2 is a compression member. The member shortens ΔL= 8*0,5/2EA= 2/EA.

Fig.5.
The normal force diagram

Fig.6.
The separated joints. On the joints act member end forces as large as but opposite directed to those of fig. 3b and 4b.

The assumption for the reaction forces at the clamps 1 and 3 is to the right, RH1 and RH3.

Σ hor. joint 1 = 0

RH1+10=0 ⇒ RH1=-10 kN.

Σ hor. joint 3 = 0

RH3+8=0 ⇒ RH3= -8 kN.

A negativ answer for bot reactions, thus not as assumed directed to the right but to the left. The joint 2 is in equilibrium, 18-10-8= 0.
Voor beide reacties een negatief antwoord, dus

Fig.7.
The construction is in equilibrium.

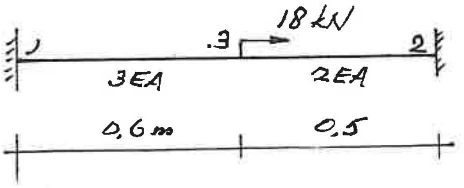


Fig.8.

$$\begin{bmatrix} F13= \\ F31= \end{bmatrix} = EA \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \cdot \begin{bmatrix} U1 \\ U3 \end{bmatrix} \qquad \begin{matrix} L=1 \\ H=3 \end{matrix}$$

F13= EA(5*U1 -5*U3)
F31= EA(-5*U1 5*U3)

$$\begin{bmatrix} F23= \\ F32= \end{bmatrix} = EA \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \cdot \begin{bmatrix} U2 \\ U3 \end{bmatrix} \qquad \begin{matrix} L=2 \\ H=3 \end{matrix}$$

F23= EA(4*U2 -4*U3)
F32= EA(-4*U2 4*U3)

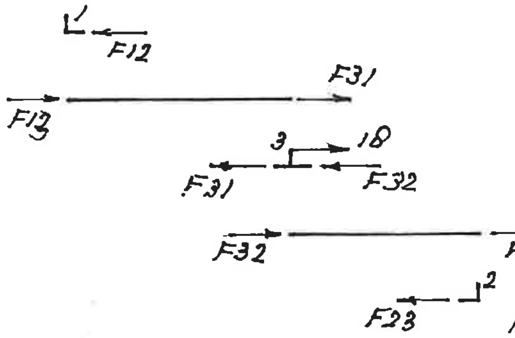


Fig.9.

$$EA \begin{bmatrix} 5 & 0 & -5 \\ 0 & 4 & -4 \\ -5 & -4 & 4+5 \end{bmatrix} \cdot \begin{bmatrix} U1 \\ U2 \\ U3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 18 \end{bmatrix}$$
$$\qquad CC \qquad\qquad \underline{u} \qquad \underline{f}$$

$$EA \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix} \cdot \begin{bmatrix} U1 \\ U2 \\ U3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 18 \end{bmatrix}$$
$$\qquad CC \qquad\qquad \underline{u} \qquad \underline{f}$$

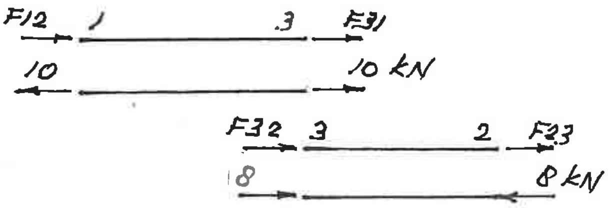


Fig.10.

Fig.8.
The same construction but with other joint numbering. So now U1=0 and U2=0. The member numbering is the same, with R1=5EA and R2=4EA.
The joint losd forces are
F1= 0 kN,   F2= 0 kN and F3= 18 kN.

The earlier found relation between member end forces and displacements can be represented as follows with L as lowest member end number and H as highest member end number.

$$\begin{bmatrix} FLH= \\ FHL= \end{bmatrix} = EA \begin{bmatrix} R & -R \\ -R & R \end{bmatrix} \cdot \begin{bmatrix} UL \\ UH \end{bmatrix}$$

FLH= EA(5*UL -5*UH)
FHL= EA(-5*UL 5*UH)

Fig.9.
The separated members and joints. The member end forces are drawn with the assumed directions from left to right. On the separated joints act member end forces as large as but opposite directed.

F13   =EA(5*U1+   -5*U3)

F23   =EA(4*U2-4*U3)

F31+F32=EA(-5*U1+0*U2+<u>5</u>*U3 +0*U1-4*U2+<u>4</u>*U3)

       =EA(-5*U1-4*U2+9*U3)

$$\begin{bmatrix} F12 \\ F23 \\ F31+F32 \end{bmatrix} = EA \begin{bmatrix} 5 & 0 & -5 \\ 0 & 4 & -4 \\ -5 & -4 & 9 \end{bmatrix} \cdot \begin{bmatrix} U1 \\ U2 \\ U3 \end{bmatrix}$$
$$\underline{f} \qquad\qquad CC \qquad\qquad \underline{u}$$

Matrix CC is symmetric with respect to the main diagonal, left to to right bottom. Member matrices S5 are symmetric as well.

Σ hor. joint 3 =0
F31+F32-18=0   or   F31+F32=18 kN
See CC * <u>u</u> = <u>f</u> shown on the left.
<u>U1=0 and U2=0</u>, rows and columns 1 and 2 are filled with zeros but on the main diagonal a <u>1</u> for CC(1,1)=0 and CC(2,2)=0. See the second relation CC * <u>u</u> = <u>f</u>. There is one equation left to solve 0*U1 +0*U2 +9*U3 =18 so that

EA*9*U3=18   from which   <u>U3=2/EA</u> like was found on page 4 .

Calculation of the member end forces..
Fig.10.
Member 1.
F13=EA(5*0-5*2/EA)   F13=-10 kN
F31=EA(-5*0+5*2/EA)   F31= 10 kN
Member 2.
F23=EA(4*0-4*2/EA)   F23= -8 kN
F32=EA(-4*0+4*2/EA)   F32= 8 kN
The member end forces are drawn with their real directions. Same result as for the same construction as on the precedin page.

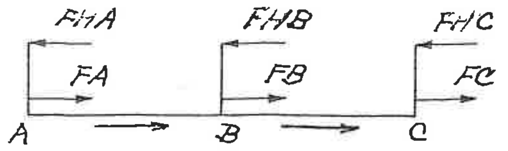


Fig.1.

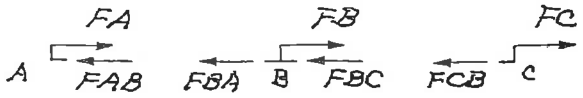


Fig.2.

$$\begin{bmatrix} FAB \\ FBA+FBC \\ FCB \end{bmatrix} \qquad \begin{bmatrix} FA \\ FB \\ FC \end{bmatrix}$$

$\underline{f}$

Fig.3.

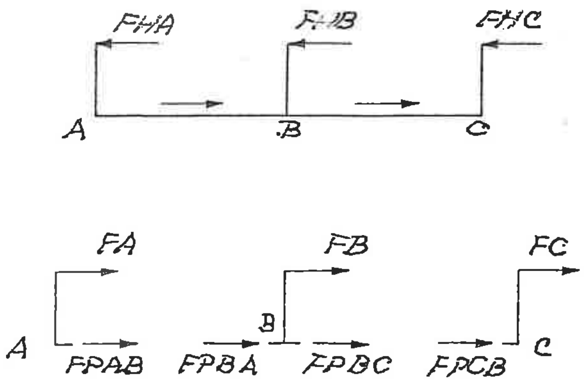
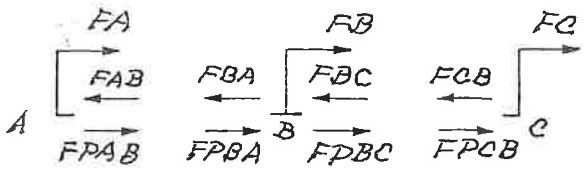


Fig.4.



$$\begin{bmatrix} FAB \\ FBA+FBC \\ FCB \end{bmatrix} \qquad \begin{bmatrix} FA+FPAB \\ FB+FPBA+FPBC \\ FC+FPCB \end{bmatrix}$$

$\underline{f}$

Fig.5.

1.3. Joint load forces and hold forces.

Fig.1.
The construction consists of two members and three joints. In unloaded state the joints A, B and C are hold with the 'hold forces' FHA, FHB and FHC. Assumed directions to the left.
Next the joint load forces FA, FB and FC are applied, assumed directions to the right, and the member load forces assumed to the right.

Fig.2.
If the joints are let loose then there are no 'hold forces'any more. The members deform and the joints displace due to the joint load forces. At the member end arise member end forces, assumed direction to the right. On the joints act forces as large as but opposite directed, thus to tghe left.
Fig.3.
If there are only joint load forces then the force vector $\underline{f}$ is filled with them as shown on page 4/6. Next the unknown joint displacements are calculated.

Fig.4.
After that the influence of the member load forces. One more time holding the joints in unloaded state, applying the loads and letting loose the joints. At the member ends arise member end forces, assumed direction to the left, acting on the joints as large as but opposite directed, thus to the right, FPAB, FBPA, FPBC and FPCB.
These forces are called primary forces, forces due to member load forcse and are calculated as the reactions of member clamped at both ends.

Fig.5.
The drawing shows the member end forces FAB, FBA, FBC and FCB being the forces due to joint load forces and member load forces.
The elements of force vecor $\underline{f}$ follow again from equilibrium of the joints.

In the drawing the member end forces now are FAB, FBA, FBC and FCB the forces due to the joint load forces and the member load forces.
The elements of force vector $\underline{f}$ follow with equilibrium of the joints.

Σ hor. joint A = 0
FA+FPBA-FAB=0 $\Rightarrow$ FAB=FA+FPBA

Σ hor. joint B = 0
FB+FPBA+FPBC-FBA-FBC=0 $\Rightarrow$ FBA+FBC=FB+FPBA+FPBC

Σ hor. joint C = 0
FC+FPCB-FCB=0 $\Rightarrow$ FCB=FC+FPCB

This way force vecor $\underline{f}$ is filled with the joint load forces and the primary forces due to the member load forces.
Remark.
The assumed directions of the forces, to the left or to the right, is arbitrary. If chosen the given way then when programming consequently applied. But, one more time, the choice of direction is arbitrary, no prescribed way!

Example.

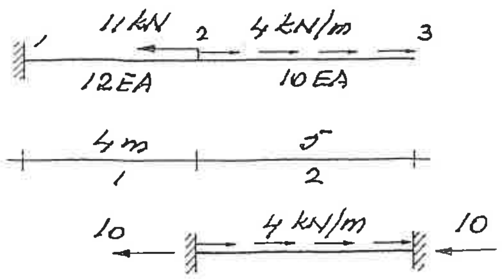


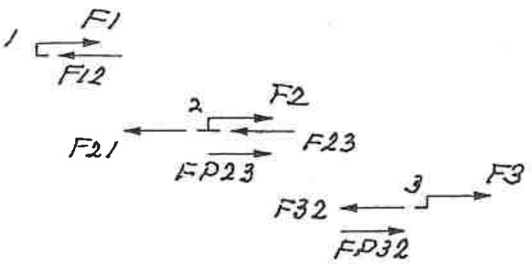
Fig.1.

Fig.1.
The construction consists of two members 1 and 2 and three joints 1, 2 and 3.
The member stiffness factors are
R1=EA1/L1=12EA/4=3EA and
R2=EA2/L2=10EA/5=2EA.

The joint load forces are
F1=0 kN, F2=-11 kN and F3=0 kN.
Member 2 is loaded with a uniformly distributed load of 4 kN/m along the member axis, to the right. The reactions of the on both ends holded member are (5*4)/2=10 kN. They are directed to the left. On the joints act forces as large as but opposite directed, to the right like the assumed direction of the primary forces
FP23=10 kN and FP32=10 kN.

$$\begin{bmatrix} F12 \\ F21 \end{bmatrix} = EA \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \cdot \begin{bmatrix} U1 \\ U2 \end{bmatrix}$$

$$\begin{bmatrix} F23 \\ F32 \end{bmatrix} = EA \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} U2 \\ U3 \end{bmatrix}$$



$$\underset{\underline{f}}{\begin{bmatrix} F12 \\ F21+F23 \\ F32 \end{bmatrix}} = EA \underset{CC}{\begin{bmatrix} 3 & -3 & 0 \\ -3 & 3+2 & -2 \\ 0 & -2 & 2 \end{bmatrix}} \cdot \underset{\underline{u}}{\begin{bmatrix} U1 \\ U2 \\ U3 \end{bmatrix}}$$

Fig.2.

Fig.2 en 3.
The elements of force vecor $\underline{f}$ follow with
Σ hor. = 0 of the joints..

Σ hor. joint 1 = 0
F1-F12=0 $\Rightarrow$ <u>F12</u>=F1=0 kn

Σ hor. joint 2 2 = 0
F2+FP23-F21-F23=0
$\Rightarrow$ <u>F21+F32</u>=F2+FP23=-11+10=-1 kN

Σ hor. joint 3 = 0
F3+FP32-F32=0 $\Rightarrow$ <u>F32</u>=F3+FP32=0+10=10 kN

$$EA \underset{CC}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & -2 \\ 0 & -2 & 2 \end{bmatrix}} \cdot \underset{\underline{u}}{\begin{bmatrix} U1 \\ U2 \\ U3 \end{bmatrix}} = \underset{\underline{f}}{\begin{bmatrix} 0 \\ -1 \\ 10 \end{bmatrix}}$$

Fig.3.

Fig.3.
The displacement of joint 1 is prescribed, is known, U1=0. Then first row and first column of construction matrix CC are filled with zeros except the element on the main diagonal, becoming CC(1,1)=1. See page 6 .

The first element of force vector $\underline{f}$ is zero because F1=0. Multiplication of the first row of CC by vector $\underline{u}$ gives $\underline{1}$*U1+0*U2+0*U3=0, and is U1=0.
Since U1=0 the first equation falls off. Two equations remain with two equations with the unknown joint displacements U2 and U3.

EA(5*U2-2*U3)= -1   2)
EA(-2*U2+2*U3)= 10 +   3)

EA(3*U2) = 9 so that U2=3/EA.

With equation 2 then follows
EA(5*3-2*U3)=-1 or EA(-2*U3)=-16
so that U3=8/EA.

The answers of U2 and U3 are positive, the joints then displace to the right as assumed. Next the member end forces F12, F21, F23 and F32 can be calculated. Next page.
The construction of figure 1 is statically determinated. The reaction at clamp 1 then is simple to be calculated. Suppose reaction RH1 is assumed to the right then follows with
Σ hor. = 0 RH1-11+4*5=0 so that RH1= -9 kN, not as assumed to the right but to the left.

1.4. Calculation of the member end forces.

$$\begin{bmatrix} F12 \\ F21 \end{bmatrix} = EA \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \cdot \begin{bmatrix} U1 \\ U2 \end{bmatrix}$$

Fig.4a.

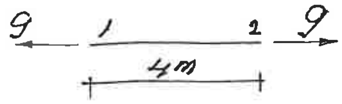


Fig.4b.

Fig.4a.
With U1=0/EA and U2=3/EA follow the member end forces with 'row times column' for member 1

F12=EA(3*0/EA-3*3/EA)= -9 kN, and
F21=EA(-3*0/EA+3*3/EA)= 9 kN.

These are member ned forces due to the displacements <u>alone</u>! Since member 1 has no member load forces are F12= -9 kN and F21= 9 kN the final member end forces.

Fig.4b.
The member end forces how they rally act at the member ends.
A negative answer for F12 so not directed as assumed to the right but to the left, and a positive answer for F21 so assumed directed to the right.

$$\begin{bmatrix} F23 \\ F32 \end{bmatrix} = EA \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} U2 \\ U3 \end{bmatrix}$$

Fig.5a.

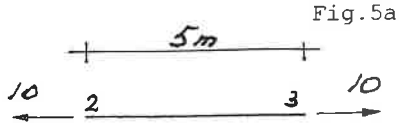


Fig.5b.

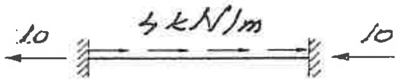


Fig.5c.

Fig.5d.

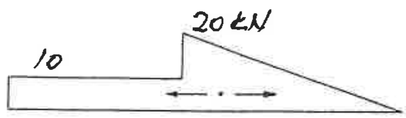


Fig.6.

Fig.5a.
Member 2 with U2=3/EA and U3=8/EA.

F23=EA(2*3/EA-2*8/EA)=-10 kN, and
F32=EA(-2*3/EA+2*8/EA)= 10 kN.

These are member end forces due to the displacements <u>alone</u>!

Fig.5b.
The member end forces how they really act at the member ends, drawn with their real directions.

Fig.5c.
The to the left directed hold forces due to the meber load forces <u>alone</u>, of the at both ends clamped member.

Fig.5d.
The final member end forces as addition of fig.5b due to joint displacements <u>alone</u>, and fig.5c due to member loads <u>alone</u>.
At member end 2 a force of 20 kN pulling at the member end and at member end 3 a force 0 kN.

Fig.6.
The normal force diagram, both members ension members.

$$\underset{\underline{f}}{\begin{bmatrix} F12 \\ F21+F23 \\ F32 \end{bmatrix}} = EA \underset{CC}{\begin{bmatrix} 3 & -3 & 0 \\ -3 & 5 & -2 \\ 0 & -2 & 2 \end{bmatrix}} \cdot \underset{\underline{u}}{\begin{bmatrix} U1 \\ U2 \\ U3 \end{bmatrix}}$$

Fig.7.

Fig.7.
With the now known joint displacements the elements of force vecor <u>f</u> can be calculated with the unchanged construction matrix CC.

These are the socalled <u>joint forces</u> K1, K2 and K3 due to the joint displacements <u>alone</u>! with assumed direction to the left.

'EA' skipped over for a while,

K1=F12 = 3*0 -3*3 +0*0= 0 -9 +0= -9 kN

K2=F21+F23=-3*0 +5*3 -2*8= 0+15-16= -1 kN

K3=F32 = 0*0 -2*3 +2*8= 0 -6+16= 10 kN

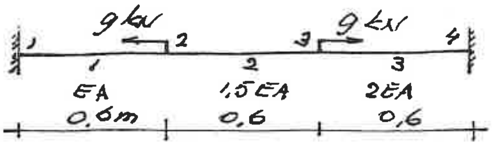


Fig.1.

member 1

$$\begin{array}{cccccc} & & & \quad 1 \qquad\quad 2 & & \\ \begin{matrix}1\\ \\2\end{matrix} & \begin{bmatrix}F12\\ \\F21\end{bmatrix} & = & \begin{bmatrix}167 & -167\\ & \\-167 & \underline{167}\end{bmatrix} & \cdot & \begin{bmatrix}U1\\ \\U2\end{bmatrix}\end{array}$$

x EA/100

member 2

$$\begin{array}{cccccc} & & & \quad 2 \qquad\quad 3 & & \\ \begin{matrix}2\\ \\3\end{matrix} & \begin{bmatrix}F23\\ \\F32\end{bmatrix} & = & \begin{bmatrix}\underline{250} & -250\\ & \\-250 & \underline{250}\end{bmatrix} & \cdot & \begin{bmatrix}U2\\ \\U3\end{bmatrix}\end{array}$$

member 3

$$\begin{array}{cccccc} & & & \quad 3 \qquad\quad 4 & & \\ \begin{matrix}3\\ \\4\end{matrix} & \begin{bmatrix}F34\\ \\F43\end{bmatrix} & = & \begin{bmatrix}\underline{333} & -333\\ & \\-333 & 333\end{bmatrix} & \cdot & \begin{bmatrix}U3\\ \\U4\end{bmatrix}\end{array}$$

$$\begin{array}{cccccc} & \quad 1 \qquad 2 \qquad 3 \qquad 4 & & & & \\ \begin{matrix}1\\2\\3\\4\end{matrix} & \begin{bmatrix}167 & -167 & . & .\\-167 & \underline{417} & -250 & .\\ . & -250 & \underline{583} & -333\\ . & . & -333 & 333\end{bmatrix} & \cdot & \begin{bmatrix}U1\\U2\\U3\\U4\end{bmatrix} & = & \begin{bmatrix}0\\-9\\9\\0\end{bmatrix}\end{array}$$

x EA/100 CC

$$\begin{array}{cccccc} & \quad 1 \qquad 2 \qquad 3 \qquad 4 & & & & \\ \begin{matrix}1\\2\\3\\4\end{matrix} & \begin{bmatrix}1 & 0 & 0 & 0\\0 & 417 & -250 & 0\\0 & -250 & 583 & 0\\0 & 0 & 0 & 1\end{bmatrix} & \cdot & \begin{bmatrix}U1\\U2\\U3\\U4\end{bmatrix} & = & \begin{bmatrix}0\\-9\\9\\0\end{bmatrix}\end{array}$$

x EA/100

$$\underbrace{\begin{bmatrix}F12\\F21+F23\\F34+F43\\F43\end{bmatrix}}_{\underline{f}} = \underbrace{\begin{bmatrix}167 & -167 & 0 & 0\\-167 & 417 & -250 & 0\\0 & -250 & 583 & -333\\0 & 0 & -333 & 333\end{bmatrix}}_{CC} \cdot \underbrace{\begin{bmatrix}U1\\U2\\U3\\U4\end{bmatrix}}_{\underline{u}}$$

Example.

Fig.1.
The construction consists of three members and four joints, regularly numbered.
Length L1, L2 and L3 is 0,6 m, strain stiffnesses EA1= 1,0EA EA2=1,5EA and EA3=2,0EA.
The joint load forces are
F2= -9 kN, directed to the left and
F3= 9 KN, directed to the right.
The member stiffness factors are
R1= EA1/L1= 1,0EA/0,6= 1,67 EA,
R2= EA2/L2= 1,5EA/0,6= 2,50 EA and
R3= EA3/L3= 2,0EA/0,6= 3,33 EA.

On the left the relation $\underline{f}$ = S5 * $\underline{u}$ of the three members is shown. Written out they are three times two equations with matching two unknown displacements, U1 and U2, U2 and U3, and U3 and U4.
Put together it gives four equations with the four unknowns shown on the left in matrix form, $\underline{f}$ = CC * $\underline{u}$.

With horizontal equilibrium of joint 2 and 3, see page 8 , follow the values of the force vector -9 kN and 9 kN.
Since the displacements of joint 1 and 4 are prescribed, are known, U1=0 and U4=0, there are two equations left with the unknowns U3 and U4.

```
EA( 4,17*U2-2,50*U3= -9,00   2)
EA(-2,50*U2+5,83*U3=  9,00   3)
```

2) times 2,50/4,17 gives

```
EA( 2,50*U2-1,50*U3= -5,40   2')    3)+2') gives
            4,33*U3=  3,60   thus  U3= 0,83/EA
next U2=-1,66/EA  with eq. 2), 3) of 2').
```

After that with the equations for each member the member end forces are calculated.

F12= 2,77 kN	F23=-6,23 kN	F34= 2,76 kN
F21=-2,77 kN	F32= 6,23 kN	F43=-2,76 kN

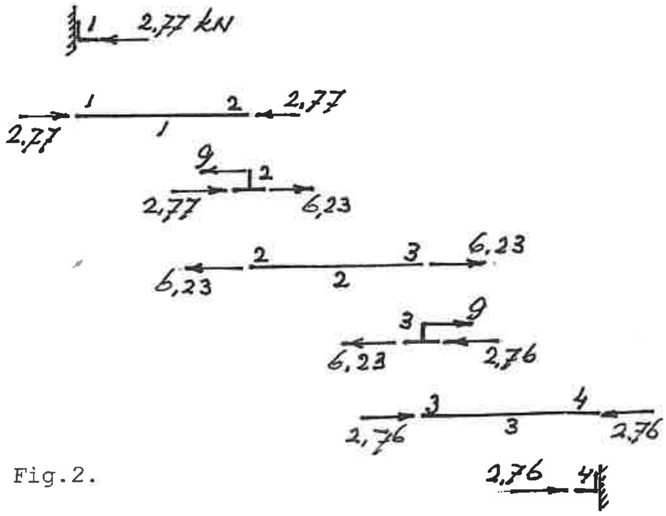


Fig.2.

Above the on the joints and member ends acting force are drawn with their real directions, and the joint load forces.
With Σ hor.= 0 for the joints 1 and 4 follow the reactions.

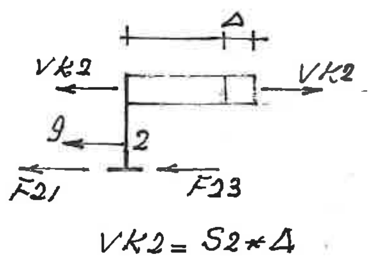


Fig.3.

Σ hor. joint 2=0 so that
F21+F23+VK2-9=0 of F21+F23+VK2= 9

Member 1
F21= EA(-167*U1 +167*U2)/100

Member 2
F23= EA(250*U2 -250*U3)/100

VK2= EA(130*U2)/100 S2=1,3EA

F21+F23+VK2= EA(-167*U1+547*U2-250*U3)

$$\begin{array}{c} \\ 1\\2\\3\\4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 547 & -250 & 0 \\ 0 & -250 & 583 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \cdot \begin{bmatrix} U1 \\ U2 \\ U3 \\ U4 \end{bmatrix} = \begin{bmatrix} 0 \\ -9 \\ 9 \\ 0 \end{bmatrix}$$

x EA/100

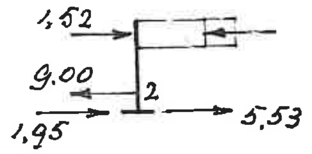


Fig.4.

$$\begin{array}{c} \\ 1\\2\\3\\4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 547 & -250 & 0 \\ 0 & -250 & 673 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \cdot \begin{bmatrix} U1 \\ U2 \\ U3 \\ U4 \end{bmatrix} = \begin{bmatrix} 0 \\ -9 \\ 9 \\ 0 \end{bmatrix}$$

x EA/100

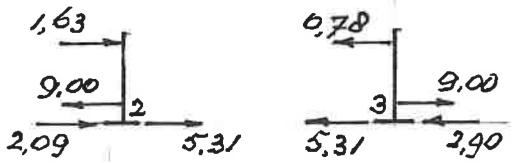


Fig.5.

Fig.3.
Joint 2 is springy supported, see page 14.
The spring constant is S2= 1,3EA kN/m.

If the spring is stretched with distance U2 with assumed direction to the right then the joint pulls at the spring with a force VK2 directed to the right. The spring pulls at the joint with a force as large as but opposite to the left directed force VK2.
The on the joint acting member end forces F21 and F23 are directed to the left as well because acting on the member ends as assumed directed to the right. further there is a joint load force of 9 kN.
With horizontal equilibrium of joint 2 and the written out equations of F21 and F23 follows (417+130)*U2= 547*U2.
The spring constant is added to the concerning element of construction matrix CC like shown here below.

$$\begin{bmatrix} F12 \\ F21+F23+VK2 \\ F32+F34 \\ F43 \end{bmatrix} = \begin{bmatrix} 167 & -167 & 0 & 0 \\ -167 & 547 & -250 & 0 \\ 0 & -250 & 583 & -333 \\ 0 & 0 & -333 & 333 \end{bmatrix} \begin{bmatrix} U1 \\ U2 \\ U3 \\ U4 \end{bmatrix}$$

f x EA/100 CC u

Like on the preceding page the unknown displacements U2 and U3 remain to be solved with
EA(5,47*U2-2,50*U3= -9 and
EA(-2,50*U2+5,83*U3= 9 from which follow

U2=-1,17/EA and U3= 1,04/EA.

And with them finally the member end forces
F12= 1,95 kN F23=-5,53 kN F34= 3,46 kN
F21=-1,95 kN F32= 5,53 kN F43=-3,46 kN

Σ hor. joint 2=0 F21+F23+VK2+9=0 or
F21+F23+VK2= -9

F21+F23+VK2= EA((5,47*(-1,17/EA)-2,50*1,04/EA)
= -6,40-2,60= -9 kN as expected.

-1,95-5,53+VK2=-9,00 so that VK2=-1,52 kN.

A negative answer, so not directed to the left on the joint as assumed but to the right.
Also is VK2= S2 * U2 =
= 1,3*EA*(-1,17/EA)=-1,52 kN

Fig.4.
The forces like they act at joint 2 drawn with their real directions.
Σ hor. joint 2=0 ? 9,00-1,95-5,53-1,52=0 OK

Fig.5.
Suppose joint 3 is springy supported as well, S3= 0,9EA kN/m. Then the concerning diagonal element of CC becomes 583+90=673 x EA/100.

Calculation gives U2=-1,25/EA and U3=0,87/EA.

F21= 2,09 kN F23= 5,31 kN VK2=-1,63 kN
F32= 5,31 kN F34= 2,90 kN VK3= 0,78 kN

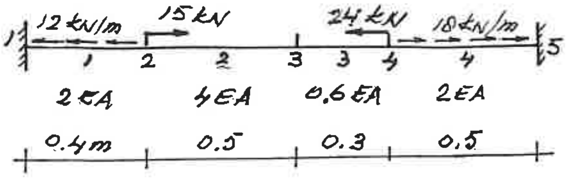


Fig.1.

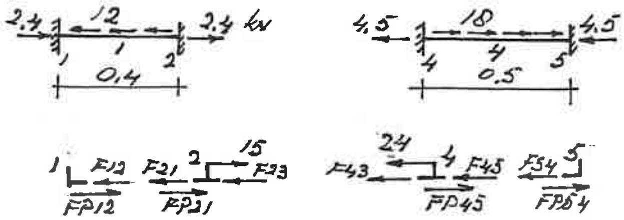


Fig.2.

F12=EA(5*U1-5*U2) F23=EA(8*U2-8*U3)
F21=EA(-5*U1+5*U2) F32=EA(-8*U2+8*U3)

F34=EA(2*U3-2*U4) F45=EA(4*U4-4*U5)
F43=EA(-2*U3+2*U4) F54=EA(-4*U4+4*U5)

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 5 & -5 & . & . & . \\ -5 & 5+8 & -8 & . & . \\ . & -8 & 8+2 & -2 & . \\ . & . & -2 & 2+4 & -4 \\ . & . & . & -4 & 4 \end{bmatrix}\end{array} \cdot \begin{bmatrix} U1 \\ U2 \\ U3 \\ U4 \\ U5 \end{bmatrix} = \begin{bmatrix} -2,4 \\ 12,6 \\ 0 \\ -19,5 \\ 4.5 \end{bmatrix}$$

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & . & . & . & . \\ . & \underline{13} & -8 & . & . \\ . & -8 & \underline{10} & -2 & . \\ . & . & -2 & \underline{6} & . \\ . & . & . & . & 1 \end{bmatrix}\end{array} \cdot \begin{bmatrix} U1 \\ U2 \\ U3 \\ U4 \\ U5 \end{bmatrix} = \begin{bmatrix} 0 \\ 12,6 \\ 0 \\ -19,5 \\ 0 \end{bmatrix}$$

times EA CC $\underline{u}$ $\underline{f}$

Fig.3.

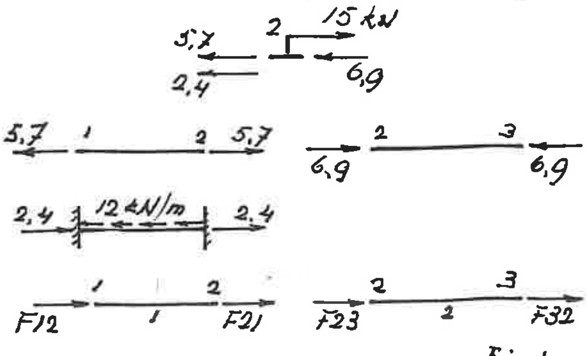


Fig.4.

Example.

Fig.1.
The stiffness factor of members 1 to 4.
R1= 2EA/0,4= 5EA, R2=4EA/0,5= 8EA,
R3=0,6EA/0,3= 2EA and R4=2EA/0,5= 4EA.
(To simplify, 'EA' now and then omitted.)

The primary forces due to the uniformly distributed loads.
Fig.2.
Member 1 with 12kN/m directed to the left.
The reactions of the at both ends clamped member are directed to the left,
(12*0,4)/2=2,4 kN.
Member 4 with 18 kN/m directed to the right.
Clamped like member 1 are the reactions directed to the right, member ends 2 and 3,
(18*0,5)/2=4,5 kN.
On the separated joints act forces as large as but opposite directed, 2,4 kN to the left and 4,5 kN to the right.
The on the separated joints acting primary forces are assumed to the right so that, page 7,
FP12=-2,4 kN and FP21=-2,4 kN,
FP45= 4,5 kN and FP54= 4,5 kN.

Construction matrix CC is composed with member stiffness matrices S5. The member end numbers determine the place where the concerning elements of S5 arrive in CC. Below S5 of member 2.

$$\begin{array}{c} \\ 2\\3 \end{array}\begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix}\end{array}$$

F23= EA(8*U2 -8*U3)
F32=-EA(-8*U2 +8*U3)

Fig.3.
The elements of force vector $\underline{f}$.

1) F12+2,4=0 or F12= -2,4 kN
2) F21+F23+2,4-15=0 or F21+F23= 12,6 kN
3) F32+F34=0 or F32+F34= 0 kN
4) F43+F45+24-4,5=0 or F43+F45=-19,5 kN
5) F54-4,5=0 or F54= 4,5 kN

With the prescribed displacements U1=0 and U5=0 follows the second construction matrix CC, equation 1) and 5) are of no use, remain

2) EA(13*U2 -8*U3)= 12,6
3) EA(-8*U2 +10*U3 -2*U4)= 0
4) EA(-2*U3 +6*U4)=-19,5

The three equations solved give
U2= 1,14/EA, U3= 0,28/EA, U4= -3,16/EA.

Fig.4.
Joint 2 in equilibrium? For the members 1 and 2 the member end forces due to the displacements alone are
F12=EA(5*U1-5*U2)=EA(5*0-5*1,14/EA)= -5,7 kN
F21=EA(-5*U1+5*U2)= 5,7 kN

F23=EA(8*U2-8*U3)=EA(8*1,14/EA-8*0,28/EA)
=9,1-2,2= 6,9 kN
F32=EA(-8*U2+8*U3)= -6,9 kN

Due to member loads alone 2,4 kN to be added.
On the left forces acting on member ends and joints drawn with their real directions.

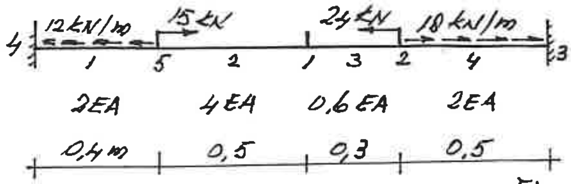


Fig.5.

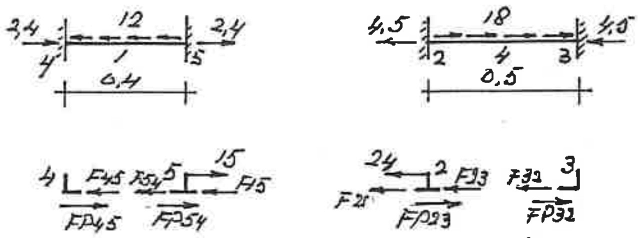


Fig.6.

F15=EA(8*U1-8*U5)
F51=EA(-8*U1+8*U5)

F12=EA(2*U1-2*U2)
F21=EA(-2*U1+2*U2)

F23=EA(4*U2-4*U3)
F32=EA(-4*U2+4*U3)

F45=EA(5*U4-5*U5)
F54=EA(-5*U4+5*U5)

$$\begin{array}{cc} & \begin{array}{ccccc}1 & 2 & 3 & 4 & 5\end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix} 8+2 & -2 & . & . & -8 \\ -2 & 2+4 & -4 & . & . \\ . & -4 & 4 & . & . \\ . & . & . & 5 & -5 \\ . & . & . & 5 & 5+8 \end{bmatrix} \end{array} \cdot \begin{bmatrix} U1 \\ U2 \\ U3 \\ U4 \\ U5 \end{bmatrix} = \begin{bmatrix} 0 \\ -19,5 \\ 4,5 \\ -2,4 \\ 12,6 \end{bmatrix}$$

$$\begin{array}{cc} & \begin{array}{ccccc}1 & 2 & 3 & 4 & 5\end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix} 10 & -2 & . & . & -8 \\ -2 & 6 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ -8 & . & . & . & 13 \end{bmatrix} \end{array} \cdot \begin{bmatrix} U1 \\ U2 \\ U3 \\ U4 \\ U5 \end{bmatrix} = \begin{bmatrix} 0 \\ -19,5 \\ 0 \\ 0 \\ 12,6 \end{bmatrix}$$

times EA CC $\underline{u}$ $\underline{f}$

Fig.7.

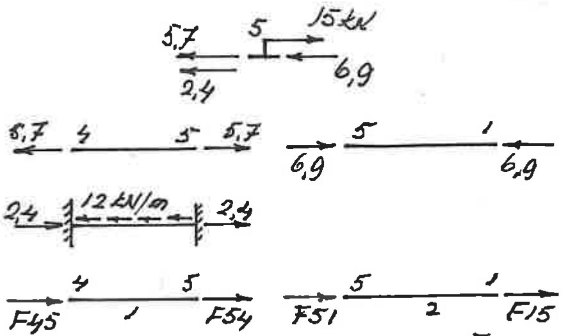


Fig.8.

Fig.5.
The same example with the same member numbering 1 to 5 from left to right. An arbitrary joint numbering, 4 5 1 2 3 from left to right.
Stiffness factors again
R1=5EA, R2=8EA, R3=2EA and R4=4EA.
(To simplify 'EA' now and then omitted.)

The primary forces due to the member loads.
Fig.6.
Member 1 with primary forces 2,4 kN on member ends 4 and 5 directed to the right.
Member 4 with primary forces 4,5 kN on member ends 2 and 3 directed to the left.

On the separated joints act forces as large as but opposite directed, 2,4 kN to the left on the joints 4 and 5, and 4,5 kN to the right on the joints 2 and 3.

The on the joints acting primary forces due to the member loads are assumed to the right so that, see page 7, note the member end numbering,
FP45=-2,4 kN and FP54=-2,4 kN,
FP23= 4,5 kN and FP32= 4,5 kN.

Construction matrix CC is composed with the member stiffness matrices S5. Here below S5 of member 2 with member end numbers 1 and 5.

$$\begin{array}{cc} & \begin{array}{cc}1 & 5\end{array} \\ \begin{array}{c}1\\5\end{array} & \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix} \end{array}$$

F15= EA(8*U1 -8*U5)
F51=-EA(-8*U1 +8*U5)

Fig.7.
The elements of force vector $\underline{f}$.

1) F15+F12=0 or F15+F12= 0 kN
2) F21+F23+24-4,5=0 or F21+F23= -19,5 kN
3) F32-4,5=0 or F32= 4,5 kN
4) F45+2,4=0 or F45= -2,4 kN
5) F54+F51+2,4-15=0 or F54+F51= 12,6 kN

With the prescribed displacements U4=0 and U3=0 follows the second matrix CC, equation 3) and 4) are of no use, are omitted, remain
1) EA(10*U1-2*U2 -8*U5)= 0
2) EA(-2*U1+6*U2)= -19,5
5) EA(-8*U1 +13*U5)= 12,6

The equations solved (with computer Gauss) give
U1= 0,28/EA, U2= -3,16/EA, U5= 1,14/EA.

Fig.8.
Is joint 5 in equilibrium? For the members 1 and 2 follow the member end forces
F45=EA(5*U4-5*U5)=EA(5*0-5*1,14/EA)= -5,7 kN
F54=EA(-5*U4+5*U5)= 5,7 kN

F15=EA(8*U1-8*U5)=EA(8*0,28/EA-8*1,14/EA)
=(2,2-9,1)= -6,9 kN
F51=EA(-8*U1+8*U5)= 6,9 kN

Joint 5 is in equilibrium.
The directions of the member end forces is assumed to the right regardless, ofcourse, the order of the member end numbering.The on the joint acting primary fores are assumed to the right as well.

1.5. The springy support.

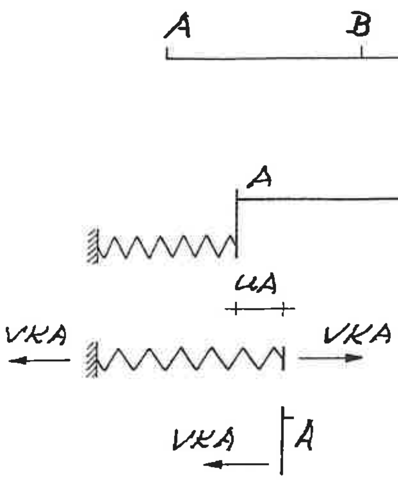


Fig. 1a.

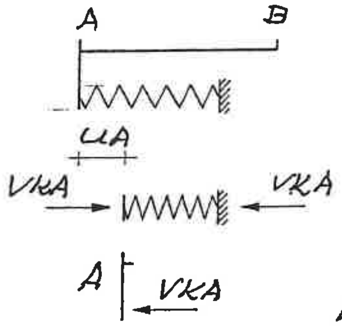


Fig. 1b.

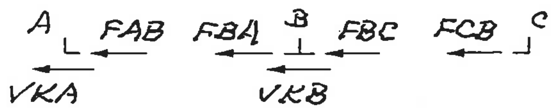


Fig. 2.

$$\underbrace{\begin{bmatrix} \text{FAB+VKA} \\ \text{FBA+FBC+VKB} \\ \text{FCB} \end{bmatrix}}_{\underline{f}} = \underbrace{\begin{bmatrix} \text{R1+SA} & \text{-R1} & \\ \text{-R1} & \text{R1+R2+SB} & \text{-R2} \\ & \text{-R2} & \text{R2} \end{bmatrix}}_{CC} \cdot \underbrace{\begin{bmatrix} \text{UA} \\ \text{UB} \\ \text{UC} \end{bmatrix}}_{\underline{u}}$$

Fig.1a.
Joint A is horizontally supported by a spring. The spring drawn here causes joint A to displace ovewr UA to the right, the assumed direction for joint displacements. The spring is stretched out. The separated spring is in equilibrium, see the two forces VKA acting at the spring ends.

At the separated spring A itself acts a force VKA as large as but opposite directed, so directed to the left.
With spring constant SA in kN/m follows spring force VKA= SA*UA kN.

Fig.1b.
In the represented case here the spring is pushed in by the assumed joint displacement UA to the right. The spring is pushed in by the forces VKA. On the separated joint A itself acts the spring force VKA as large as but opposite directed, so directed to the left.

So that in both schematic represented cases 1a and 1b with the assumed joint displacement UA to the right the spring force VKA acting on the separated joint is directed to the left.

Fig.2.
With the assumed spring supports of the joints A and B with spring constants SA and SB act on the separated joints A and B spring forces VKA=SA*UA and VKB=SB*UB directed to the left due to the assumed joint displacements UA and UB to the right.
The member end forces assumed directed to the right act on the joints as large as but opposite directed.

Force vector f consists of (see page 3) the unknown to be calculated forces FAB and VKA, FBA, FBC and VKB, and FCB.

```
FAB=  R1*UA -R1*UB        FBA= R2*UB -R2*UC
FBA= -R1*UA +R1*UB        FBC=-R2*UB +R2*UC

FAB+VKA      =  R1*UA+SA*UA -R1*UB
             = (R1+SA)*UA   -R1*UB

FBA+FBC+VKB= -R1*UA         +R1*UB
             +R2*UB+SB*UB -R2*UC

             = -R1*UA +(R1+R2+SB)*UB

FCB          = -R2*UB +R2*UC
```

all together represented on the left in matrix form. The spring constants SA and SB are added to the concerning elements on the main diagonal of the construction matrix CC.

The spring constants SA and SB have the same dimension like the stiffness factors R1=EA1/L1 and R2=EA2/L2.
R2=EA2/L2.
'R=EA/L' with (kN/m^2)(m^2)/m or kN/m and SA and SB in kN/m.

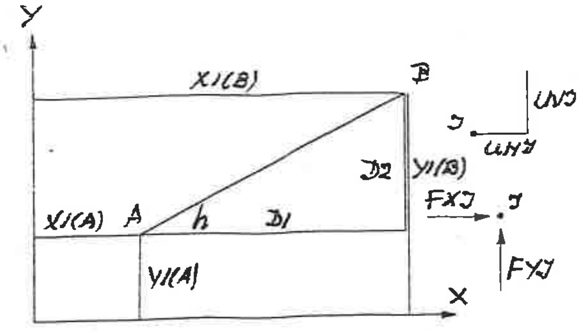


Fig.1a.

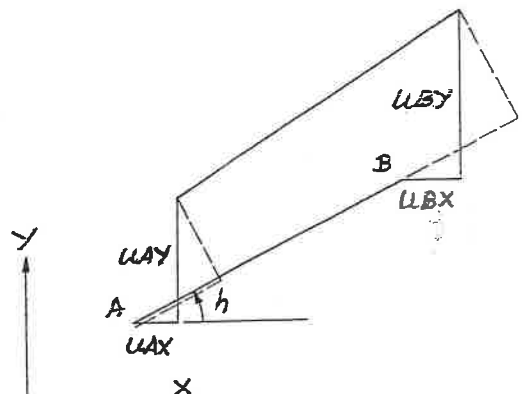


Fig. 1b.

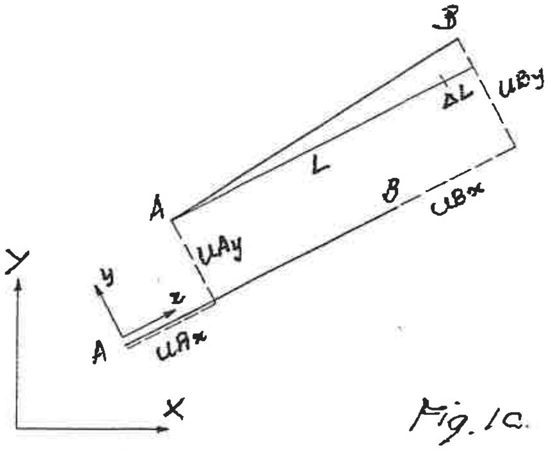


Fig.1c.

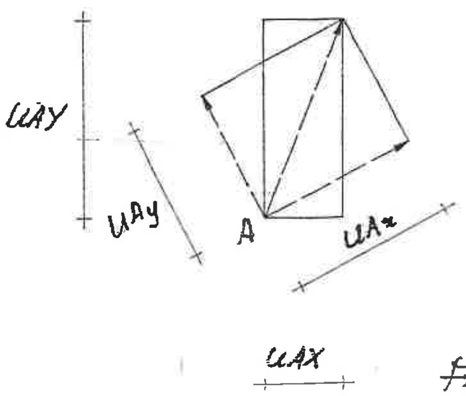


Fig.2a.

2. Plane trusses of which the joints are regarded as hinges

2.1. The ralation between member end forces and member end displacements, being the joint displacements.

Fig.1a.
Assumptions.
Yhe X-Y axis system (capitals) is the construction axis system. Starting point is the drawn member AB.

The horizontal displacement UHI of joint I is assumed to the right like the X axis, the vertical displacement UV1 upward like the Y axis (not because it should be like that).

On the joint act horizontal joint load forces FX1 assumed to the right and vertical joint load forces FY1 assumed upward.
(The vertical joint load forces are mostly directed downward, could have been assumed also.)

It is assumed that the coordinates X1(B) and Y1(B) of member end B are larger than X1(A) and Y1(A) of member end A. Then the triangle lengths are

D1=X1(B)-X1(A) and D2=Y1(B)-Y1(A),
and member lengyh L1=Sqr(D1^2+D2^2).

Further are Sin(h)=D2/L1 or S=D2/L1 and
Cos(h)=D1/L1 or C=D1/L1.

Fig. 1b en 1c.
The displacements UAX and UAY of the member ends are assumed to the right, UAY and UBY upward.
The member itself has an own member axis system x-y of which the origin is assumed at member end A. The x axis is directed from A to B, the y axis perpendicular to AB like drawn.

(Later the joints and thus also the member ends are numbered. Then A represents the lowest member end number L and B the highest member end number H. The member axis system itself then is always placed at the lowest member end number L.)

If one assumes that displacement UBx of member end B is larger than UAx of member end A then the member gets ΔL longer.
Since the displacements UAy and UBy perpendicular to the member axis are small with respect to member length L one can write

ΔL= UBx - UAx.

Fig.2a.
The displacements UAx and UAy w.r.t. the member axis system x-y will be expressed in the displacements UAX and UAY w.r.t. the construction axis system X-Y.
Because the member end forces in X and Y direction will be expressed in the displaments w.r.t. the construction axis system X-Y by means of member stiffness matrix S5.

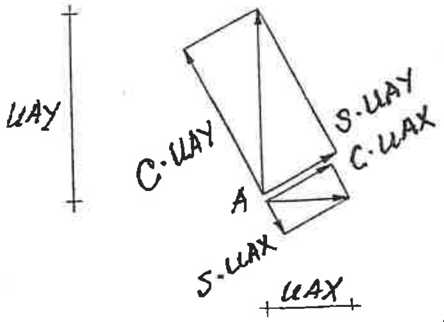


Fig.2b.

The displacements UAx and UAy of member end A.

Fig. 2a and 2b.
The like vectors drawn displacements UAX and UAY are rseolved into a displacement along and a displacement perpendicular to the member axis

The component of UAX alog the member axis x is Cos(h)*UAX, or C*UAX with C=Cos(h), and the component of UAY along the member axis is Sin(h)*UAY, or S*UAX with S=Sin(h).
Then the displacement UAx (small x, not X), fig.2a, along the x-axis is
UAx= C*UAX + S*UAY.

Perpendicular to axis x are the components
Sin(h)*UAX or S*UAX, and
Cos(h)*UAY or C*UAY.
Taking into account the directions of the components follows for displacement UAY according to member axis y

UAy= C*UAY - S*UAX or, other order,

UAy= -S*UAX + C*UAY.

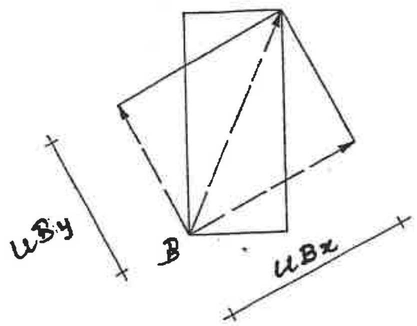


Fig.3a.

The displacements UBx and UBy of member end B.

Fig. 3a en 3b.
Like done for member end A follow

UBx= C*UBX + S*UBY, and

UBy= -S*UBX + C*UBY.

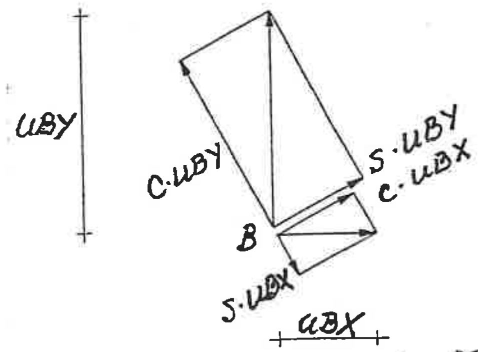


Fig.3b.

Next the relation between member end forces w.r.t. member anxis system x-y, and the member end dispacements w.r.t. construction axis system X-Y.

Fig.4.
The member gets ΔL= UBx-UAx longer, the member is a tension member. Then act on the member ends tensile forces of F kN.

With Hooke's law is ΔL=FL/EA or F=(EA/L)*ΔL.

The member stiffness factor R=(EA/L). Then is

F= R*(UBx-UAx) or F= R(-UAx+UBx).

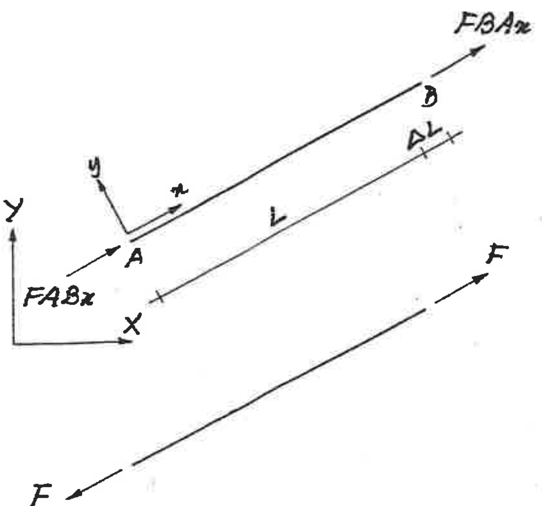


Fig.4.

The assumed direction of the member end forces FABx and FBAx is the same like for the displacements UAx and UBx, like the x axis.

Both member end forces FABx and F at A, and both member end forces FBAx and F at B, are equal, see also page / , then follows

FABx=-F so that FABx= R(UAx-UBx), and

FBAX= F so that FBAx= R(-UAx+UBx).

If the earlier found UAx and UAy are put in in both equations, follow

FABx= R((C*UAX+S*UAY)-(C*UBX+S*UBY)) and

FBAx= R(-(C*UAX+S*UAY)+(C*UBX+S*UBY)).

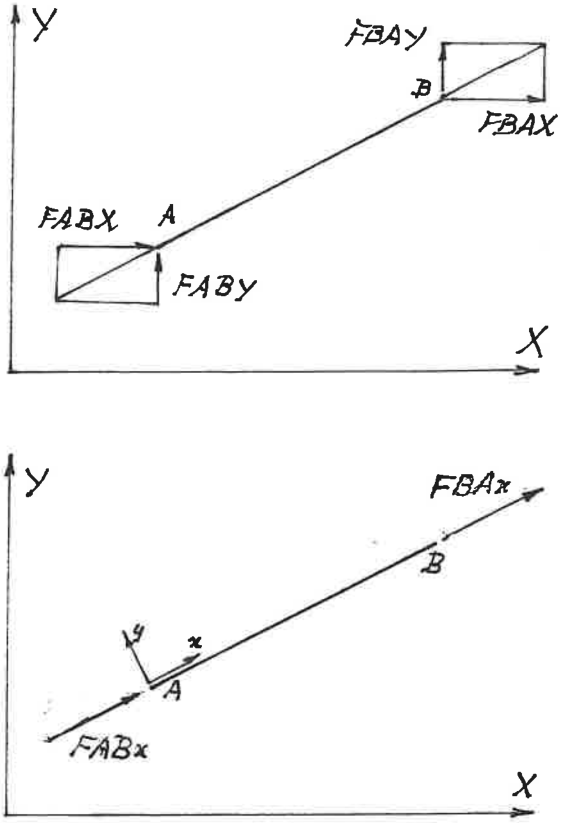


Fig. 5.

$$\left[\begin{array}{cc|cc} R*C^2 & R*S*C & -R*C^2 & -R*S*C \\ R*S*C & R*S^2 & -R*S*C & -R*S^2 \\ \hline -R*C^2 & -R*S*C & R*C^2 & R*S*C \\ -R*S*C & -R*S^2 & R*S*C & R*S^2 \end{array}\right]$$

S5

Stiffness factor R=EA/L in kN/m.

Matrix S5 has three values, three different elements, with a + or a minus - sign.

+/- R*C^2, R*S*C, R*S^2 with

C=Cos(h) en S=Sin(h).

$$\left[\begin{array}{cc|cc} o & oo & -o & -oo \\ oo & ooo & -oo & -ooo \\ \hline -o & -oo & o & oo \\ -oo & -ooo & oo & ooo \end{array}\right]$$

The four sub matrices are symmetric as well w.r.t. the main diagonal.

Fig.5.
The assumption for the directions of the horizontal member end forces FABX and FBAX is chosen like for the horizotal displacements UAX and UBX to the right, and the assumption for the direction of the vertical member end forces FABY and FBAY is like for the vertical displacements UAY and UBY upward. These member end forces are the components of FABx and FBAx.

Cos(h)=FABX/FABx	so that	FABX=FABx*Cos(H)
Sin(h)=FABY/FABx		FABU=FABx*Sin(h)
Cos(H)=FBAX/FBAx	so that	FBAX=FBAx*Cos(h)
Sin(h)=FBAY/FBAx		FBAY=FBAx*Sin(h)

With C=Cos(h) and S=Sin(h) then follow

at A <u>FABX=FABx*C</u> 1) and <u>FABY=FABx*S</u> 2) and

at B <u>FBAX=FBAx*C</u> 3) and <u>FBAY=FBAx*S</u> 4).

The equations for the in accordance with the x axis acting member end forces FABx and FBAx, expressed in the member end displacements w.r.t. the construction axis system X-Y, UAX and UBX, UBX and UBY, were found on the preceding page,

<u>FABx= R((C*UAX+S*UAY)-(C*UBX+S*UBY))</u> and

<u>FBAx= R(-(C*UAX+S*UAY)+(C*UBX+S*UBY))</u>.

If they are put in the equations of the horizontal and vertical member end forces, FABX and FABY, FBAX and FBAY, then follow

FABX= R(C*C*UAX +S*C*UAY -C*C*UBX -S*C*UBY) 1)

FABY= R(S*C*YAX +S*S*UAY -S*C*UBX -S*S*UBY) 2)

FBAX= R(-C*C*UAX -S*C*UBY +C*C*UBX +S*C*UBY) 3)

FBAY= R(-S*C*UAX -S*S*UAY +S*C*UBX +S*S*UBY) 4)

and written in matrix form

$$\underset{\underline{f}}{\begin{bmatrix} FABX \\ FABY \\ FBAX \\ FBAY \end{bmatrix}} = \underset{S5}{\begin{bmatrix} R*C*C & R*S*C & -R*C*C & -R*S*C \\ R*S*C & R*S*S & -R*S*C & -R*S*S \\ -R*C*C & -R*S*C & R*C*C & R*S*C \\ -R*S*C & -R*S*S & R*S*C & R*S*S \end{bmatrix}} \cdot \underset{\underline{u}}{\begin{bmatrix} UAX \\ UAY \\ UBX \\ UBY \end{bmatrix}}$$

<u>f</u> is the force vector,

S5 is the member stiffness matrix, and

<u>u</u> is the displacement vector.

Matrix S5 is symmetric w.r.t. the main diagonal from left top to right bottom.

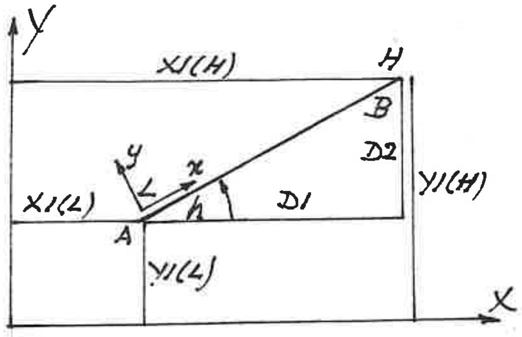


Fig. 1.

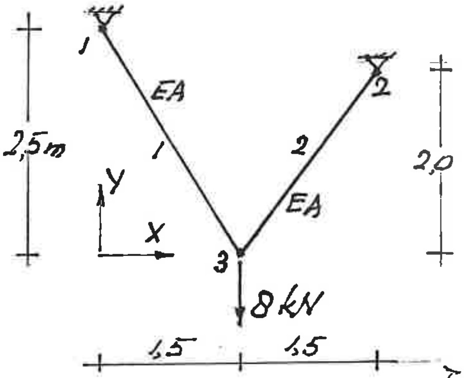


Fig. 2.

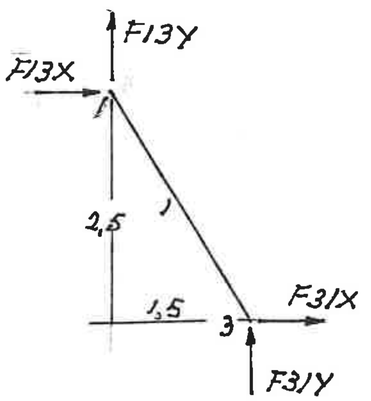


Fig. 3.

Here above member 1 with member end forces and member end displacements with their assumed directions. Here below their written out equations.

F13X= 90UH1 -150UV1 -90UH3 +150UV3

F13Y=-150UH1 +251UV1 +150UH3 -251UV3

F31X= -90UH1 +150UV1 +90UH3 -150UV3

F31Y= 150UH1 -251UV1 -150UH3 +251UV3

kN = kN/m times m.

Example.

Fog.1.
Assumptions.
Lowest member end number L instead of letter A with coordinates X1(L) and Y1(L) and highest member end number H instead of letter B with coordinates X1(H) and Y1(H).

<u>D1=X1(H)-X1(L)</u> and <u>D2=Y1(H)-Y1(L)</u>.

For D1 and D2, the coordinaat with the highest member end number H minus the coordinate with the lowest member end number L.

Member length is Sqr(D1^2+D2^2).

Fig.2.
Member 1 and member 2 with strain stiffness EA.

X1(1)= 0	X1(2)= 3,0	X1(3)= 1,5	m
Y1(1)= 2,5	Y1(2)= 2,0	Y1(3)= 0	m

<u>The member stiffness matrix of member 1.</u>

Fig.2 en 3.
Member end numbers H=3 and L=1.

D1=X1(3)-X1(1)= 1,5-0 = 1,5 D1= 1,5
D2=Y1(3)-Y1(1)= 0 -2,5=-2,5 D2=-2,5

L1= Sqr((1,5)^2+(-2,5)^2)=Sqr(8,50)= 2,92 m

Stiffness factor R1= 'EA/L'= EA/2,92= 0,342 EA.

Modulus of elasticity E,
E=210*10^3 N/mm^2 = 210 kN/mm2 =210*10^6 kN/m^2
Member cross section A in m^2 then follows EA is E times A, with E in kN/m^2 and A in m^2, so that <u>EA in kN</u>.

R1='EA/L' in kN/m, then the elements of S5 are with C and S, in kN/m if the values of E in kN/m^2 and A in m^2 are brought in the calculation.

Cos(h) is C= D1/L1= 1,5/2,92= 0,514
Sin(h) is S= D2/L1= -2,5/2,92= -0,856

Next the three combinations of R1, C en S.

R1*C^2= 0,342*(0,514)^2 = 0,090 EA
R1*S*C= 0,342*(-0,856)*0,514= -0,150 EA
R1*S^2= 0,342*(-0,856)^2 = 0,251 EA

For the sake of convenience elements of S5 are multiplied by 1000 and divided by EA. Then follows <u>f</u> = S5 * <u>u</u> like found on the preceding page.

$$\begin{bmatrix} F13X \\ F13Y \\ F31X \\ F31Y \end{bmatrix} = \left[\begin{array}{cc|cc} 90 & -150 & -90 & 150 \\ -150 & 251 & 150 & -251 \\ \hline -90 & 150 & 90 & -150 \\ 150 & -251 & -150 & 251 \end{array}\right] \cdot \begin{bmatrix} UH1 \\ UV1 \\ UH3 \\ UV3 \end{bmatrix}$$

<u>x EA/1000</u>

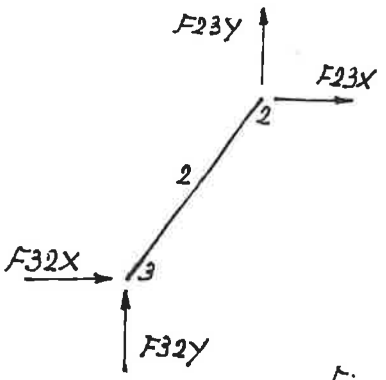


Fig. 4.

F23X= 144UH2 +192UV2 -144UH3 -192UV3

F23Y= 192UH2 +256UV2 -192UH3 -256UV3

F32X=-144UH2 -192UV2 +144UH3 +192UV3

F32Y=-192UH2 -256UV2 +192UH3 +256UV3

$$\begin{bmatrix} F13X \\ F13Y \\ . \\ . \\ F31X \\ F31Y \end{bmatrix} = \begin{bmatrix} 90 & -150 & . & . & -90 & 150 \\ -150 & 251 & . & . & 150 & -251 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ -90 & 150 & . & . & 90 & -150 \\ 150 & -251 & . & . & -150 & 251 \end{bmatrix}$$

Both member stiffness matrices are combined to the construction stiffness matrix CC like shown here.
See also page 10.

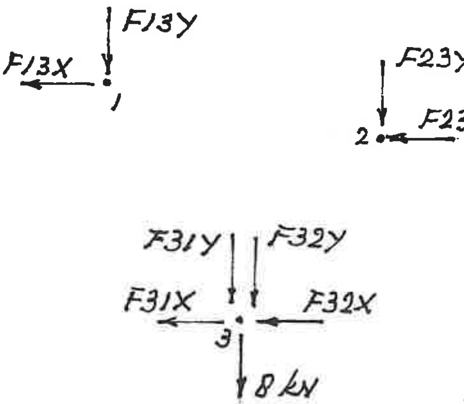


Fig. 5.

On the joints act member end forces as large as but opposite directed. The equilibrium of the joints deliver the elements of f, see on the right, all zero except the last element,

Σ vert. joint 3=0 F31Y+F32Y+8=0 so that F31Y+F32Y= -8 kN.

The member stiffness matrix of member 2.

Fig. 2 en 4.
Member end numbers H=3 and L=2.
D1=X1(3)-X1(2)= 1,5-3,0=-1,5 D1=-1,5 m
D2=Y1(3)-Y1(2)= 0 -2,0=-2,0 D2=-2,0 m

L1=Sqr((-1,5)^2+(-2,5)^2)=Sqr(6,25)= 2,50 m
Stiffness factor R2= 'EA/L'= EA/2,50= 0,400 EA.

C=D1/L1= -1,5/2,50= -0,600
S=D2/L1= -2,0/2,50= -0,800

R2*C^2= 0,400*(-0,600)^2 = 0,144 EA
R2*S*C= 0,400*(-0,800)*(-0,600)= 0,192 EA
R2*S^2= 0,400*(-0,800)^2 = 0,256 EA

$$\begin{bmatrix} F23X \\ F23Y \\ F32X \\ F32Y \end{bmatrix} = \left[\begin{array}{cc|cc} 144 & 192 & -144 & -192 \\ 192 & 256 & -192 & -256 \\ \hline -144 & -192 & 144 & 192 \\ -192 & -256 & 192 & 256 \end{array}\right] \cdot \begin{bmatrix} UH2 \\ UV2 \\ UH3 \\ UV3 \end{bmatrix}$$

x EA/1000

$$\begin{bmatrix} . \\ . \\ F23X \\ F23Y \\ F32X \\ F32Y \end{bmatrix} = \begin{bmatrix} . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & 144 & 192 & -144 & -192 \\ . & . & 192 & 256 & -192 & -256 \\ . & . & -144 & -192 & 144 & 192 \\ . & . & -192 & -256 & 192 & 256 \end{bmatrix} \cdot \begin{bmatrix} UH1 \\ UV1 \\ UH2 \\ UV2 \\ UH3 \\ UV3 \end{bmatrix}$$

$$\begin{bmatrix} F13X \\ F13Y \\ F23X \\ F23Y \\ F31X+F32X \\ F31Y+F32Y \end{bmatrix} = \begin{bmatrix} 90 & -150 & . & . & -90 & -150 \\ -150 & 251 & . & . & 150 & -251 \\ . & . & 144 & 192 & -144 & -192 \\ . & . & 192 & 256 & -192 & -256 \\ -90 & 150 & -144 & -192 & 234 & 42 \\ 150 & -251 & -192 & -256 & 42 & 507 \end{bmatrix}$$

CC

$$\underbrace{\begin{bmatrix} 90 & -150 & . & . & -90 & -150 \\ -150 & 251 & . & . & 150 & -251 \\ . & . & 144 & 192 & -144 & -192 \\ . & . & 192 & 256 & -192 & -256 \\ -90 & 150 & -144 & -192 & 234 & 42 \\ 150 & -251 & -192 & -256 & 42 & 507 \end{bmatrix}}_{CC} \cdot \underbrace{\begin{bmatrix} UH1 \\ UV1 \\ UH2 \\ UV2 \\ UH3 \\ UV3 \end{bmatrix}}_{\underline{u}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -8 \end{bmatrix}}_{\underline{f}}$$

These are the equations in matrix form which have to be solved.

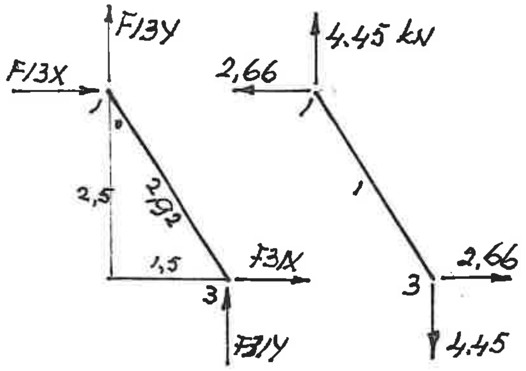


Fig.6a. Fig.6b.

See the equations of the member end forces of member 1, page .
With UH1=0 and UV1=0 is, without EA,
F13X=-0,090*UH3 +0,150*UV3 =
=-0,090*2,87 +0,150*(-16,02)=

F13X= -0,26 -2,40 = -2,66 kN

F13Y= 0,150*2,87 -0,251*(-16,02)=

F13Y= 0,43 +4,02 = 4,45 kN

In similar way one finds
Zo vinft men op dezelfde wijze

F31X= 2,66 kN and F31Y= -4,45 kN

Fig.6b.

The member end forces drawn with their real directions.

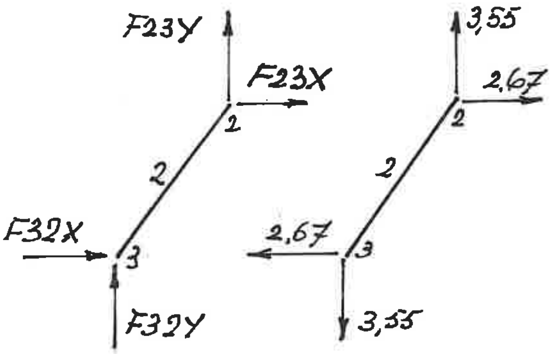


Fig.7a en 7b.

With the equations given on the preceding page follow

F23X= 2,67 kN and F23Y= 3,55 kN
F32X= -2,67 kN and F32Y= -3,55 kN

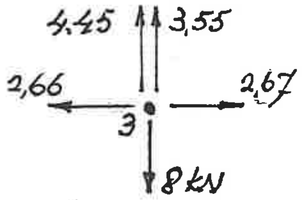


Fig.8.

On joint 3 act member end forces as large as but opposite directed.
Joint 3 is in equilibrium.

There are now six equations with six unknowns but four are already known, UH1=0, UV(1)=0, UH(2)=0 and UV(2)=0.
(In a computer program the number of unknowns can stay the same if the necessary changes are applied like on page . The concerning rows and columns are filled with zeros and the diagonal elements become 1. The system of six equations is then solved the Gaus-, Crout- or Inverse-method.)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \underline{234} & \underline{42} \\ 0 & 0 & 0 & 0 & \underline{42} & \underline{507} \end{bmatrix} \begin{bmatrix} UH1 \\ UV1 \\ UH2 \\ UV2 \\ UH3 \\ UV3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -8 \end{bmatrix}$$

CC x EA/1000 $\underline{u}$ $\underline{f}$

For e.g. UH2 then follows (EA omitted)
0*UH1+0*UV1+<u>1*UH2</u>+0*UV2+0*UH3+0*UV3= 0 UH2=0.

The solution of the two remaining equations can be as follows.

0,234*UH3 +0,042*UV3 = 0 5)

0,042*UH3 +0,507*UV3 = -8 x (0,234/0,042)

0,234*UH3 +2,825*UV3 = -44,57 6)

<u>-2,783*UV3 = 44,57</u> 5) minus 6)

UV3= 44,57/(-2,783) = -16,02 <u>UV3= -16,02/EA</u>

and with UV3 then follows <u>UH3 = 2,87/EA</u>.

Joint 3 displaces downward and to the right.

Member 1 is a tensile member with a tensile force Sqr(2,66^2+4,45^2)=Sqr(26,88)= <u>5,18 kN</u> which lengthens the member.

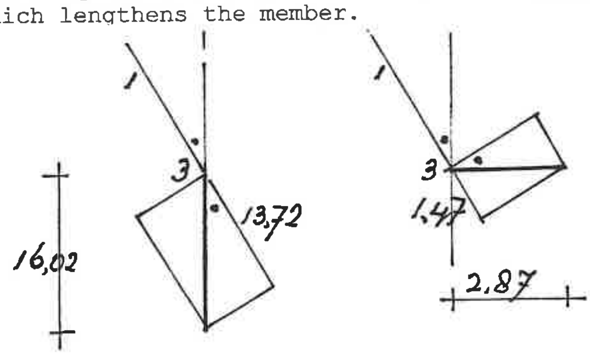


Fig.9.

UV3=-16,02 downward can be resolved along and perpendicular to the member.
Along the member (16,02/2,92)*2,5= <u>13,72 /EA</u>.

UH3= 2,87 to the right can be resolved in similar way
Along the member (2,87/2,92)*1,5= <u>1,47 /EA</u>.

Member 1 lenghtens 13,72/EA+1,47/EA= 15,19/EA.
ΔL=FL/EA is 15,19/EA=F*2,92/EA from which
F=15,19/2,92= 5,20 kN 'is' 5,18 kN, correct.

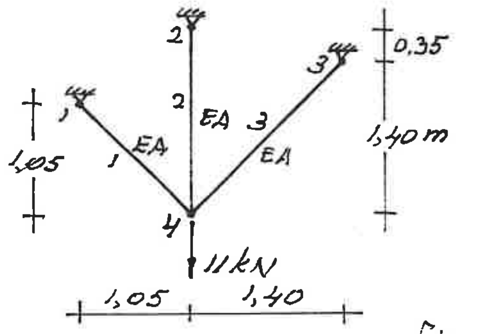


Fig.1.

$$
\begin{array}{c|c}
 & \begin{array}{cccccccc} 1 & 2 & & & & & 7 & 8 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ \\ \\ \\ \\ 7 \\ 8 \end{array} &
\left[\begin{array}{cccccccc}
334 & -334 & . & . & . & . & -334 & 334 \\
-334 & 334 & . & . & . & . & 334 & -334 \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
-334 & 334 & . & . & . & . & \underline{334} & \underline{-334} \\
334 & -334 & . & . & . & . & \underline{-334} & \underline{334}
\end{array}\right]
\end{array}
$$

member 1

$$
\begin{array}{c|c}
 & \begin{array}{cccccccc} & & 3 & 4 & & & 7 & 8 \end{array} \\
\begin{array}{c} \\ \\ 3 \\ 4 \\ \\ \\ 7 \\ 8 \end{array} &
\left[\begin{array}{cccccccc}
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
. & . & 0 & 0 & . & . & 0 & 0 \\
. & . & 0 & 571 & . & . & 0 & -571 \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
. & . & 0 & 0 & . & . & \underline{0} & \underline{0} \\
. & . & 0 & -571 & . & . & \underline{0} & \underline{571}
\end{array}\right]
\end{array}
$$

member 2

$$
\begin{array}{c|c}
 & \begin{array}{cccccccc} & & & & 5 & 6 & 7 & 8 \end{array} \\
\begin{array}{c} \\ \\ \\ \\ 5 \\ 6 \\ 7 \\ 8 \end{array} &
\left[\begin{array}{cccccccc}
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
. & . & . & . & 252 & 252 & -252 & -252 \\
. & . & . & . & 252 & 252 & -252 & -252 \\
. & . & . & . & -252 & -252 & \underline{252} & \underline{252} \\
. & . & . & . & -252 & -252 & \underline{252} & \underline{252}
\end{array}\right]
\end{array}
$$

staaf 3

Of eight equations two remain to calculate the unknowns UH4 and UV4.

The on the member ends upward acting member end forces act on joint 4 opposite directed, is downward.
The joint load force of 11 kN is also directed downward.
Σ vert. joint 4=0 F41Y+F42Y+F43Y+11=0 of F41Y+F42Y+F43Y= -11 kN.

The two equations are, times EA,

0,586*UH4 -0,082*UV4= 0

-0,082*UH4 +1,157*UV4= -11 of which

<u>UH4= -1,34/EA</u> and <u>UV4= -9,60/EA.</u>

Example.

Fig.1.

Three members, statically indeterminate. Four joints with eight displacements of which only two are unknown, UH4 and UV4.

Member 1. L1= 1,49 m R1=EA/1,49= 0,671 EA
D1= 1,05 m Cos(h) is C= 1,05/1,49= 0,705
D2=-1,05 m Sin(h) is S=-1,05/1,49=-0,705
R1*C^2= 0,671*(0,705)^2 = 0,334
R1*S*C= 0,671*(-0,705)*0,705= -0,334
R1*S^2= 0,671*(-0,705)^2 = 0,334

Member 2. L2= 1,75 m R2=EA/1,75= 0,571 EA
D1= 0 m C=0/1,75 = 0
D2=-1,75 m S=-1,75/1,75= -1
R2*C^2= 0,571*0^2 = 0,000
R2*S*C= 0,571*(-1)*0= 0,000
R2*S^2= 0,571*(-1)^2= 0,571

Member 3. L=1,98 m R3=EA/1,98= 0,505 EA
D1=-1,40 C=-1,40/1,98= -0,707
D2=-1,40 S=-1,40/1,98= -0,707
R3*C^2= 0,505*(-0,707)^2 = 0,252
R3*S*C= 0,505*(-0,707)*(-0,707)= 0,252
R3*S^2= 0,505*(-0,707)^2 = 0,252

$$
\left[\begin{array}{c} F14X \\ F14Y \\ . \\ . \\ . \\ . \\ F41X \\ F41Y \end{array}\right] = S5* \left[\begin{array}{c} UH1 \\ UV1 \\ . \\ . \\ \\ \\ UH4 \\ UV4 \end{array}\right]
\qquad
\left[\begin{array}{c} . \\ . \\ F24X \\ F24Y \\ . \\ . \\ F42X \\ F42Y \end{array}\right] = S5* \left[\begin{array}{c} . \\ . \\ UH2 \\ UV2 \\ . \\ . \\ UH4 \\ UV4 \end{array}\right]
$$

member 1 member 2

$$
\begin{array}{c|c}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} &
\left[\begin{array}{cccccccc}
334 & -334 & . & . & . & . & -334 & 334 \\
-334 & 334 & . & . & . & . & 334 & -334 \\
. & . & 0 & 0 & . & . & 0 & 0 \\
. & . & 0 & 571 & . & . & 0 & -571 \\
. & . & . & . & 252 & 252 & -252 & -252 \\
. & . & . & . & 252 & 252 & -252 & -252 \\
-334 & 334 & 0 & 0 & -252 & -252 & \underline{586} & \underline{-82} \\
334 & -334 & 0 & -571 & -252 & -252 & \underline{-82} & \underline{1157}
\end{array}\right]
\end{array}
$$

$$
\begin{array}{c|c}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} &
\left[\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \underline{586} & \underline{-82} \\
0 & 0 & 0 & 0 & 0 & 0 & \underline{-82} & \underline{1157}
\end{array}\right]
\end{array}
* \left[\begin{array}{c} UH1 \\ UV1 \\ UH2 \\ UV2 \\ UH3 \\ UV3 \\ UH4 \\ UV4 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -11 \end{array}\right]
$$

x EA/1000 CC

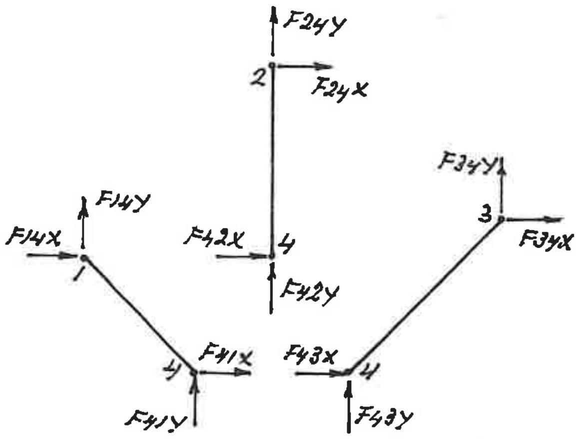


Fig.2a.

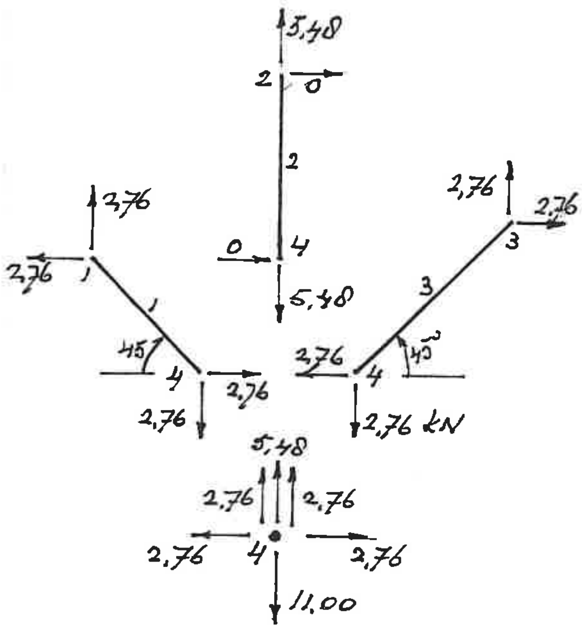


Fig.2b.

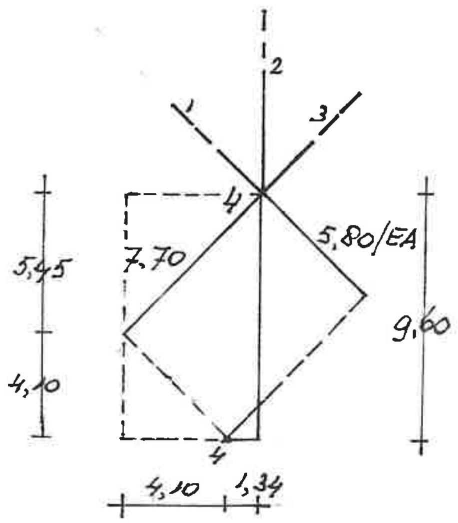


Fig. 3.

Calculation of member end forces with the help of the elements of the member stiffness matrices given in construction stiffness matrix CC, see preceding page. Zero multiplications are omitted.

Fig.2a en 2b.

F41X= EA(0,334*UH4 -0,334*UV4)
    = EA(0,334*(-1,34/EA) -0,334*(-9,60/EA)
    = -0,45 +3,21= 2,76 kN

F41Y= EA(-0,334*(-1,34/EA) +0.334*(-9,60/EA)
    = 0,45 -3,21= -2,76 kN

F42x= EA(0*UH4 +0*UV4) = 0 kN

F42Y= EA(0*UH4 +0,571*(-9,60/EA)=
    = 0 - 5,48 = -5,48 kN

F43X= EA(0,252*(-1,34/EA) +0,252*(-9,60/EA)
    = -0,34 -2,42 = -2,76 kN

A negative answer, not to the right as assumed but directed to the left.

F43Y= EA(0,252*(-1,34/EA) +0,252*(-9,60/EA)
    = -0,34 -2,42 = -2,76 kN

A negative answer, not upward as assumed but directed downward.

With Σ hor.=0 and Σ vert.=0 of the three members follow the member end forces at the member ends 1, 2 and 3.

On the separated joint 4 act member end forces as large as but opposite directed.
Sum horizontal and sum vertical of joint 4 is zero, equilibrium.

The displacements of joint 4.
Fig.3

UH4= -1,34/EA, negative answer, joint 4 does not displace to the right as assumed but to the left.

UV4= -9,60/EA, negative answer, joint 4 does not displace upward as assumed but downward.

Member 1 is a tensile member, the tensile force is 2,76*Sqr(2)= 3,89 kN.
With member length 1,49 m the member lengthens,

with ΔL='FL/EA' follows 3,89*1,49/EA = 5,80/EA.

Member 3 is a tensile force, 3,89 kN as well, with length 1,98 the member lengthens 3,89*1,98/EA = 7,70/EA.

Member 2 is a tensile member, tensile forse 5,48 kN, with a length of 1,75 m the member becomes 5,48*1,75/EA= 9,60/EA longer.

With some extra lines in the figure follows with geometry
5,80/Sqr(2)= 4,10 and 7,70/Sqr(2)= 5,45.

4,10+1,34= 5,44 'is' 5,45 en
4,10+5,45= 9,55 $\approx$ 9,60

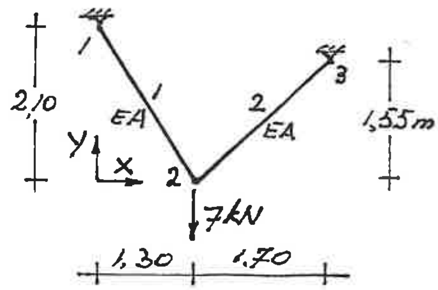


Fig.1a.

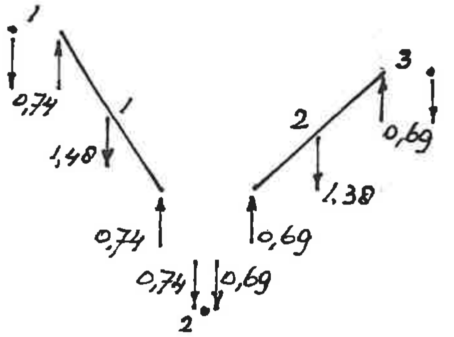


Fig.1b.

$$
\begin{bmatrix} F12X \\ F12Y \\ \\ F21X \\ F21Y \end{bmatrix} = \begin{bmatrix} 112 & -181 & -112 & 181 \\ -181 & 293 & 181 & -293 \\ \\ -112 & 181 & \underline{112} & \underline{-181} \\ 181 & -293 & \underline{-181} & \underline{293} \end{bmatrix} \begin{bmatrix} UH1 \\ UV1 \\ \\ UH2 \\ UV2 \end{bmatrix}
$$

x EA/1000

Fig.2.
The member end forces of member 1 due to the joint displacements alone!

F12X= -0,112(1,78) +0,181(-17,31)
    = -0,20 -3,13 = -3,33 kN

F12Y= 0,181(1,78) -0,293(-17,31)
    = 0,32 +5,07 = 5,39 kN

F21X= 0,20 +3,13 = 3,33 kN

F21Y= -0,32 -5,07 = -5,39 kN

The member end forces due to own weight alone are added like shown here below.

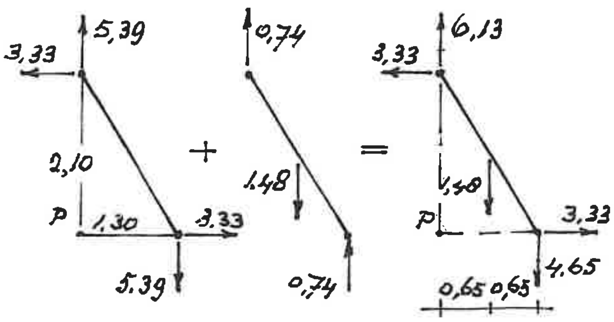


Fig.2.

Member 1 in equilibrium?

Σ mom. P= 0
1,48(0,65)-3,33(2,10)-4,65(1,30)=

0,96-6,99+6,05= 0,02 ≈ 0 OK.

Example.

Fig.1a en 1b.

Member 1. L1= 2,47 m   R1=EA/2,47= 0,405
C=0,526   S=-0,850
R1*C^2= 0,112 R1*S*C=-0,181 R1*S^2= 0,293

Member 2. L2= 2,30 m   R2=EA/2,30= 0,435
C=0,739   S= 0,674
R2*C^2= 0,238 R2*S*C= 0,217 R2*S^2=0,198

With these data the construction stiffness matrix can be composed like shown earlier.
The elements of force vector f consist of joint load forces and the joint load forces due to the member loads, here own weight of the members 0,6 kN/m. (see 1,48 and 1,38 kN)

For member 1 they are the on the joints 1 and 2 acting forces of (0,6*2,47)/2= 0,74 kN and for member 2 the on joints 2 and 3 acting forces of staaf 2 de op de knopen 2 en 3 werkende (0,6*2,30)/2=0,69 kN.

$$
\begin{bmatrix} F23X \\ F23Y \\ \\ F32X \\ F32Y \end{bmatrix} = \begin{bmatrix} \underline{238} & \underline{217} & -238 & -217 \\ \underline{217} & \underline{198} & -217 & -198 \\ \\ -238 & -217 & 238 & 217 \\ -217 & -198 & 217 & 198 \end{bmatrix} \begin{bmatrix} UH2 \\ UV2 \\ \\ UH3 \\ UV3 \end{bmatrix}
$$

The joint load forces are assumed directed upward. The vertical member end forces F12Y, F21Y, F23Y and F32Y are assumed directed upward and thus on the joints downward.
Due to own weight act on the member ends forces directed upward and thus on the joints directed downward.

Σ vert. joint 1=0 F12Y+0,78=0 F12Y=-0,78 kN
On joint 2 act 7,00 kN, 0,78 kN and 0,69 kN, together 8,43 kN.

Σ vert. joint 2=0 F21Y+F23Y +8,43=0 or
          F21Y+F23Y= -8,43 kN.
Σ vert. joint 3=0 F32Y+0,69=0 F32Y=-0,69 kN

$$
\underset{CC}{\begin{bmatrix} 112 & -181 & -112 & 181 & . & . \\ -181 & 293 & 181 & -293 & . & . \\ \\ -112 & 181 & \underline{350} & \underline{36} & -238 & -217 \\ 181 & -293 & \underline{36} & \underline{491} & -217 & -198 \\ \\ . & . & -238 & -217 & 238 & 217 \\ . & . & -217 & -198 & 217 & 198 \end{bmatrix}} \underset{\underline{u}}{\begin{bmatrix} UH1 \\ UV1 \\ \\ UH2 \\ UV2 \\ \\ UH3 \\ UV3 \end{bmatrix}} = \underset{\underline{f}}{\begin{bmatrix} 0 \\ -0,78 \\ \\ 0 \\ -8,43 \\ \\ 0 \\ -0,69 \end{bmatrix}}
$$

The unknown displacements UH2 and UV2 are calculated by solving the next two equations.

0,350*UH2 +0,036*UV2= 0

0,036*UH2 +0,491*UV2= -8,43   from which

UH2= 1,78/EA   and   UV2= -17,31/EA.

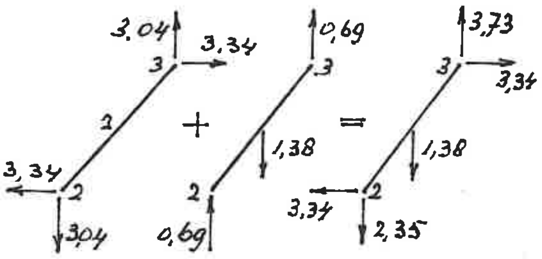


Fig. 3.

Fig.3.
The member end forces of member 2 due to the joint displacements alone.

F23X= 0,238(1,78) +0,217(-17,31)
    = -0,42 +3,76 = 3,34 kN

F23Y= 0,217(1,78) +0,198(-17,31)
    = 0,39 -3,43 = -3,04 kN

F32X= 0,42 -3,76 = -3,34 kN

F32Y= -0,39 +3,43 = 3,04 kN

The member end forces due to own weight are added.

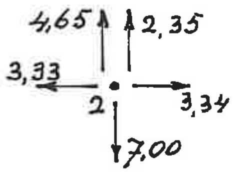


Fig. 4.

Fig.4.
The on joint 2 acting member end forces due to joint displacements and ownweight and the joint load force of 7 kN.
For joint 2 is Σ hor.=0 and Σ vert=0.

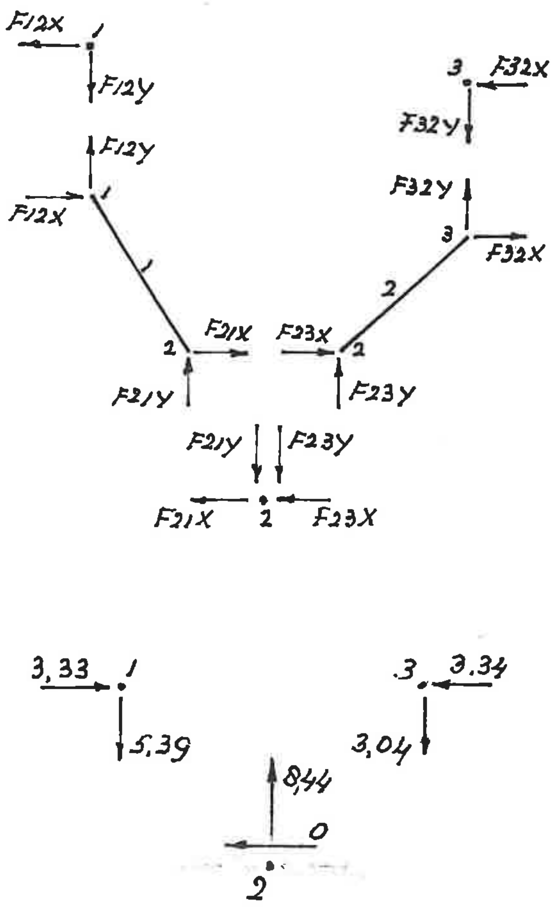


Fig. 5.

Fig.5.
The elements of force vector $\underline{f}$ here below consist of the 'sum of the on the joints acting member end forces due to the displacements'.

$$\begin{bmatrix} F12X \\ F12Y \\ F21X+F23X \\ F21Y+F23Y \\ F32X \\ F32Y \end{bmatrix} = \begin{bmatrix} 112 & -181 & -112 & 181 & . & . \\ -181 & 293 & 181 & -293 & . & . \\ -112 & 181 & \underline{350} & \underline{36} & -238 & -217 \\ 181 & -293 & \underline{36} & \underline{491} & -217 & -198 \\ . & . & -238 & -217 & 238 & 217 \\ . & . & -217 & -198 & 217 & 198 \end{bmatrix} \cdot$$

$\underline{f}$ x EA/1000   CC

$$\begin{bmatrix} 0 \\ 0 \\ 1,78 \\ -17,31 \\ 0 \\ 0 \end{bmatrix}$$

/EA $\underline{u}$

F12X= -3,33 kN
F12Y= 5,39 kN

F21X+F23X= 0   kN
F21Y+F23Y= -8,44 kN

F32X= 3,34 kN
F32Y= 3,04 kN

F12X= -0,112(1,78) +0,181(-17,31)
    = -0,20 -3,13 = -3,33 kN

F12Y= 0,181(1,78) -0,293(-17,31)
    = 0,32 +5,07 = 5,39 kN

F21X+F23X= 0,350(1,78) +0,036(-17,31)
         = 0,62 -0,62 = 0 kN

F21Y+F23Y= 0,036(1,78) +0,491(-17,31)
         = 0,06 -8,50 = -8,44 kN

F32X= -0,238(1,78) -0,217(-17,31)
    = -0,42 +3,76 = 3,34 kN

F32Y= -0,217(1,78) -0,198(-17,31)
    = -0,39 +3,43 = 3,04 kN

8,44 kN due to the displacements alone.
With own weight follows
8,44-0,74-0,69= 7,01 ≈ 7,00 ok

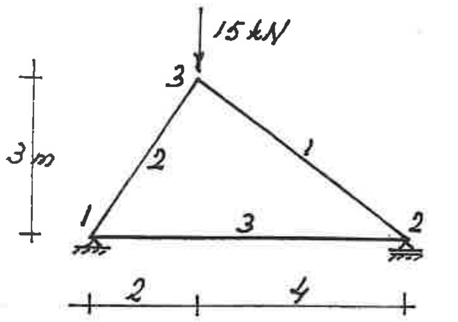


Fig. 1.

$$\begin{bmatrix} F13X \\ F13Y \\ \cdot \\ \cdot \\ F31X \\ F31Y \end{bmatrix} = \begin{bmatrix} \underline{85} & 128 & \cdot & \cdot & -85 & -128 \\ 128 & 191 & \cdot & \cdot & -128 & -191 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -85 & -128 & \cdot & \cdot & \underline{85} & \underline{128} \\ -128 & -191 & \cdot & \cdot & \underline{128} & \underline{191} \end{bmatrix}$$

member 2

$$\begin{bmatrix} F12X \\ F12Y \\ F21X \\ F21Y \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \underline{167} & 0 & -167 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot \\ -167 & 0 & \underline{167} & 0 & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

member 3

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{295} & 0 & -128 & 96 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -128 & 0 & \underline{213} & \underline{32} \\ 0 & 0 & 96 & 0 & \underline{32} & \underline{263} \end{bmatrix}}_{CC} \cdot \underbrace{\begin{bmatrix} UH1 \\ UV1 \\ UH2 \\ UV2 \\ UH3 \\ UV3 \end{bmatrix}}_{\underline{u}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -15 \end{bmatrix}}_{\underline{f}}$$

Three joints with two displacements, with 3 x 2 = 6 equations.
The displacements UH1-0, UV(1)=0 and UV(2)=0 are known. In that case a 1 on the main diagonal and zeros on the concerning rows and columns. 1*UH1-0 etc.

0,295*UH2 -0,128*UH3 +0.096*UV3= 0

-0,128*UH2 +0,213*UH3 +0,032*UV3= 0

0,096*UH2 +0.032*UH3 +0,263*UV3= -15

With computer-GAUSS page 94 follow
<u>UH2= 40,2</u> <u>UH3=35,6</u> <u>UV3=-76,0</u> /EA.

Example.

<u>Member 1</u> with length L1= Sqr(4^2+3^2)= 5,00 m.
<u>H=3 L=2</u> R1= EA/L1=EA/5,00= 0,200 EA kN/m

X1(3)=2 X1(2)=6 D1= 2-6=-4,00 m
Y1(3)=3 Y1(2)=0 D2= 3-0= 3,00 m
C=D1/L1=-4,00/5,00= -0,800
S=D2/L1= 3,00/5,00= 0,600

R1*C^2= 0,200*(-0,800)^2 = 0,128 EA
R1*S*C= 0,200*0,600*(-0,800)=-0,096 EA
R1*S^2= 0,200*(0,600)^2 = 0,072 EA

$$\begin{bmatrix} \cdot \\ \cdot \\ F23X \\ F23Y \\ F32X \\ F32Y \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 128 & -96 & -128 & 96 \\ \cdot & \cdot & -96 & 72 & 96 & -72 \\ \cdot & \cdot & -128 & 96 & 128 & -96 \\ \cdot & \cdot & 96 & -72 & -96 & 72 \end{bmatrix} \cdot \begin{bmatrix} UH1 \\ UV1 \\ UH2 \\ UV2 \\ UH3 \\ UV3 \end{bmatrix}$$

member 1

<u>Member 3</u> with length L1= 6,00 m.
<u>H=2 L=1</u> R1= EA/L1=EA/6,00= 0,167 EA

X1(2)=6 X1(1)=0 D1=X1(2)-X1(1)= 6-0= 6,00 m
Y1(2)=0 Y1(1)=0 D2=Y1(2)-Y1(1)= 0-0= 0,00 m
C=D1/L1= 6,00/6,00= 1,000
S=D2/L1= 0,00/6,00= 0,000

R1*C^2= 0,167*(1,000)^2 = 0,167
R1*C*S= 0,167*(0,000)*1,000= 0,000
R1*S^2= 0,167*(0,000)^2 = 0,000

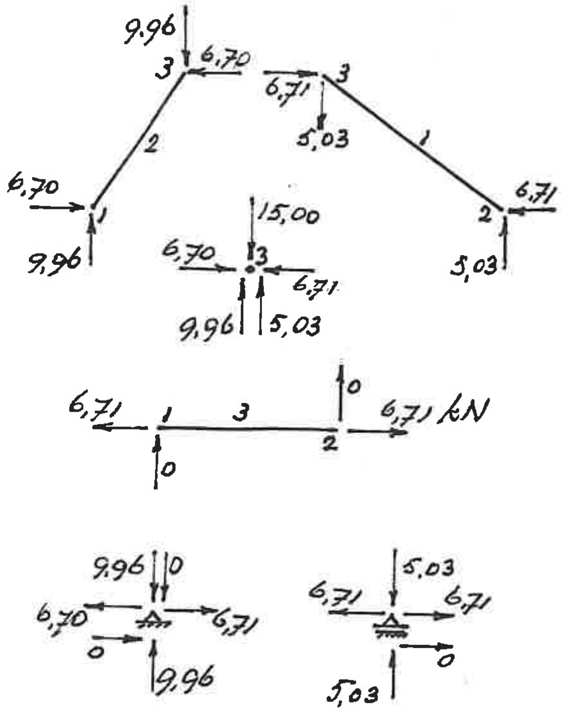
$$\begin{bmatrix} F12X+F13X \\ F12Y+F13Y \\ F21X+F23X \\ F21Y+F23Y \\ F31X+F32X \\ F31Y+F32Y \end{bmatrix} = \begin{bmatrix} \underline{252} & 128 & -167 & 0 & -85 & -128 \\ 128 & 191 & 0 & 0 & -128 & -191 \\ -167 & 0 & \underline{295} & -96 & -128 & 96 \\ 0 & 0 & -96 & 72 & 96 & -72 \\ -85 & -128 & -128 & 96 & \underline{213} & \underline{32} \\ -128 & -191 & 96 & -72 & \underline{32} & \underline{264} \end{bmatrix}$$

x EA/1000

The underlined elements are the sums of the concerning elements of the member stiffness matrices, 85UH1 + 167UH1, 85 + 167= 252, and 128UH2 + 167UH2, 128 + 167= 295.

$$\underbrace{\begin{bmatrix} \underline{252} & 128 & -167 & 0 & -85 & -128 \\ 128 & 191 & 0 & 0 & -128 & -191 \\ -167 & 0 & \underline{295} & -96 & -128 & 96 \\ 0 & 0 & -96 & 72 & 96 & -72 \\ -85 & -128 & -128 & 96 & \underline{213} & \underline{32} \\ -128 & -191 & 96 & -72 & \underline{32} & \underline{264} \end{bmatrix}}_{CC} \cdot \underbrace{\begin{bmatrix} UH1 \\ UV1 \\ UH2 \\ UV2 \\ UH3 \\ UV3 \end{bmatrix}}_{\underline{u}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -15 \end{bmatrix}}_{\underline{f}}$$

F31Y+F32Y+15=0 so that F31Y+F32Y=-15.



$$
\begin{array}{c|cccccc}
 & & & 3 & 4 & 5 & 6 \\
\hline
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
3 & \cdot & \cdot & 128 & -96 & -128 & -96 \\
4 & \cdot & \cdot & -96 & 72 & 96 & -72 \\
5 & \cdot & \cdot & -128 & 96 & 128 & -96 \\
6 & \cdot & \cdot & 96 & -72 & -96 & 72
\end{array}
$$

member 1

$$
\begin{array}{c|cccccc}
 & 1 & 2 & & & 5 & 6 \\
\hline
1 & \underline{85} & 128 & \cdot & \cdot & -85 & -128 \\
2 & 128 & 191 & \cdot & \cdot & -128 & -191 \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
5 & -85 & -128 & \cdot & \cdot & \underline{85} & \underline{128} \\
6 & -128 & -191 & \cdot & \cdot & \underline{128} & \underline{191}
\end{array}
$$

memmber 2

$$
\begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & & \\
\hline
1 & \underline{167} & 0 & -167 & 0 & \cdot & \cdot \\
2 & 0 & 0 & 0 & 0 & \cdot & \cdot \\
3 & -167 & 0 & \underline{167} & 0 & \cdot & \cdot \\
4 & 0 & 0 & 0 & 0 & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

member 3

Calculation of some member end forces with help of $\underline{f}$ = S5 * $\underline{u}$ of the members given of the preceding page. EA is omitted, the zero multiplications as well.

Member 2.
F31X= 0,085*UH3 +0,128*UV3=
0,085*35,6 +0,128*(-76,0)= 3,03 -9,73= -6,70 kN

F31Y= 0,128*UH3 +0,191*UV3=
0,128*35,6 +0,191*(-76,0)= 4,56-14,52= -9,96 kN

Similar with F31X and F31Y, or with Σ hor.=0 and Σ vert.=0 of member 2.
F31X+F13X=0 -6,70+F13X=0 F13X= 6,70 kN
F31Y+F13Y=0 -9,96+F13Y=0 F13Y= 9,96 kN

In similar way one finds
F32X= 6,71 F32Y=-5,03 F23X=-6,71 F23Y= 5,03
F12X=-6,71 F12Y= 0 F21X= 6,71 F21Y= 0

On the left the member end forces are drawn with their real directions. On the separated joints act forces as large as but opposite directed.
With Σ hor.=0 and Σ vert.=0 follow the support reactions.

Joint numbers L and H are member end numbers.

$$
\underbrace{\begin{bmatrix} FLHX \\ FLHY \\ FHLX \\ FHLY \end{bmatrix}}_{\underline{f}} = \underbrace{\begin{bmatrix} R*C*C & R*S*C & -R*C*C & -R*S*C \\ R*S*C & R*S*S & -R*S*C & -R*S*S \\ -R*C*C & -R*S*C & R*C*C & R*S*C \\ -R*S*C & -R*S*S & R*S*C & R*S*S \end{bmatrix}}_{S5} \cdot \underbrace{\begin{bmatrix} ULX \\ ULY \\ UHX \\ UHY \end{bmatrix}}_{\underline{u}}
$$

The lowest member end number L and the highest member end number H determine the place of an element of member stiffness matrix S5 in the construction stiffness matrix CC.

Row and column numbers 2*L-1, 2*L, 2*H-1, 2*H.

Member 1. F23X L=2 2*2-1= **3** and 2*2= **4**
H=3 2*3-1= **5** and 2*3= **6**
For the rows and columns 3 and 4, and 5 and 6.

$$
\begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
2 & (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
3 & (3,1) & (3,2) & \underline{(3,3)} & \underline{(3,4)} & \underline{(3,5)} & \underline{(3,6)} \\
4 & (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
5 & (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\
6 & (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)
\end{array}
$$

CC

Likewise for member 2 and 3. See on the left the elements of the three S5's placed in CC. The coinciding elements, on same places, are added.
(5,5) of member 1 + (5,5) of member 2 is 213,
(2,1) member 2 + (2,1) member 3 is 128, etc.

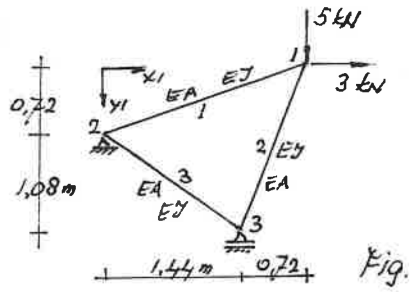


Fig.

$$
\begin{array}{cc}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{bmatrix} F12X \\ F12Y \\ F21X \\ F21Y \end{bmatrix}
&
\begin{bmatrix} \underline{394} & \underline{131} & -394 & -131 \\ \underline{131} & \underline{44} & -131 & -44 \\ -394 & -131 & \underline{394} & \underline{131} \\ -131 & -44 & \underline{131} & \underline{44} \end{bmatrix}
\cdot
\begin{bmatrix} UH1 \\ UV1 \\ UH2 \\ UV2 \end{bmatrix}
\end{array}
$$

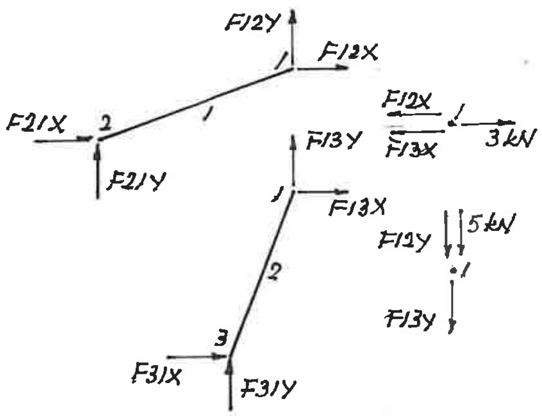
member 1

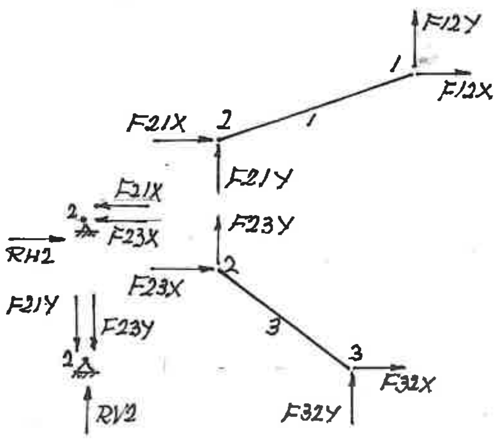
$$
\begin{array}{cc}
 & \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array}
\begin{bmatrix} F13X \\ F13Y \\ F31X \\ F31Y \end{bmatrix}
&
= \begin{bmatrix} \underline{71} & \underline{177} & -71 & -177 \\ \underline{177} & \underline{444} & -177 & -444 \\ -71 & -177 & \underline{71} & \underline{177} \\ -177 & -444 & \underline{177} & \underline{444} \end{bmatrix}
\cdot
\begin{bmatrix} UH1 \\ UV1 \\ UH3 \\ UV3 \end{bmatrix}
\end{array}
$$

member 2

$$
\begin{array}{cc}
 & \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\
\begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{bmatrix} F23X \\ F23Y \\ F32X \\ F32Y \end{bmatrix}
&
= \begin{bmatrix} \underline{356} & \underline{-267} & -356 & 267 \\ \underline{-267} & \underline{200} & 267 & -200 \\ -356 & 267 & \underline{356} & \underline{-267} \\ 267 & -200 & \underline{-267} & \underline{200} \end{bmatrix}
\cdot
\begin{bmatrix} UH2 \\ UV2 \\ UH3 \\ UV3 \end{bmatrix}
\end{array}
$$

member 3 <u>x EA/1000</u>





Example

The elements of the stiffness matrices are calculated like done on the preceding page.

$$
\underset{CC}{\begin{bmatrix} \underline{465} & \underline{308} & -394 & -131 & -71 & -177 \\ \underline{308} & \underline{488} & -131 & -44 & -177 & -444 \\ -394 & -131 & \underline{750} & \underline{398} & -356 & -267 \\ -131 & -44 & \underline{398} & \underline{244} & -267 & -200 \\ -71 & -177 & -356 & -267 & \underline{427} & \underline{-90} \\ -177 & -444 & -267 & -200 & \underline{-90} & \underline{644} \end{bmatrix}}
\cdot
\underset{\underline{u}}{\begin{bmatrix} UH1 \\ UV1 \\ UH2 \\ UV2 \\ UH3 \\ UV3 \end{bmatrix}}
=
\underset{\underline{f}}{\begin{bmatrix} 3 \\ -5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}
$$

Since the joint displacements UH2, UV2, and UV3 are known, here all three zero, UH2=0, UH2=0 and UH3=0, the concerning equations can be missed. The altered CC then looks like shown here below.

$$
\underset{\text{x EA/1000} \quad CC}{\begin{bmatrix} 465 & 308 & . & . & -71 & . \\ 308 & 488 & . & . & -177 & . \\ . & . & 1 & . & . & . \\ . & . & . & 1 & . & . \\ -71 & -177 & . & . & 427 & . \\ . & . & . & . & . & 1 \end{bmatrix}}
\cdot
\underset{\underline{u}}{\begin{bmatrix} UH1 \\ UV1 \\ UH2 \\ UV2 \\ UH3 \\ UV3 \end{bmatrix}}
=
\underset{\underline{f}}{\begin{bmatrix} \mathbf{3} \\ \mathbf{-5} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}
$$

Fig.2.
The elements of force vector $\underline{f}$ follow with equilibrium of the joints.

Σ hor. joint 1 = 0
F12X+F13X -3 = 0 F12X+F13X= **3**
Σ vert. joint 1 = 0
F12Y+F13Y +5 = 0 F12Y+F13Y= **-5**

0,465*UH1 +0,308*UV1 -0.071*UH3= 3

0,308*UH1 +0,488*UV1 -0,177*UH3= -5

0,071*UH1 -0,177*UV1 +0,427*UH3= 0

With computer Gauss follow
<u>UH1= 23,90/EA</u> <u>UV1= -28,12/EA</u> <u>UH3= -7,79/EA</u>.

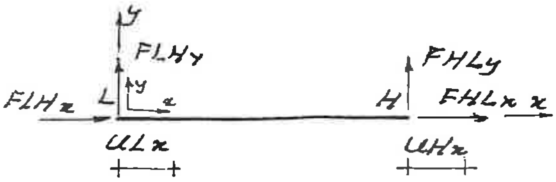
The reactions RH2 and RV2 of joint 2.
Fig.3.

F21X+F23X = see 1st and 3rd member matrix.
member 1 F21X third row times column $\underline{u}$ and
member 3 F23X first row times column $\underline{u}$.
-0,394(23,90)-0,131(-28,12)-0,356(-7,79)=-2,97

F21X+F23X -RH2=0 -2,97 -RH2=0 <u>RH2=-2,97 kN</u>
Not to the right but directed to the left.

F21Y+F23Y = see 1st and 3rd member matrix.
member 1 F21Y fourth row times column $\underline{u}$ and
member 3 F23Y second row times column $\underline{u}$.
-0,131(23,90)-0,044(-28,12)+0,267(-7,79)=-3,97

F21Y+F23Y -RV2=0 -3,97 -RV2=0 <u>RV2=-3,97 kN</u>
not directed upward as assumed but downward.



$$\begin{bmatrix} FLHx \\ FHLx \end{bmatrix} = \begin{bmatrix} R & -R \\ -R & R \end{bmatrix} \cdot \begin{bmatrix} ULx \\ UHx \end{bmatrix} \qquad R=EA/L$$

Fig.1a.

$$\begin{bmatrix} FLHy \\ FHLy \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} ULy \\ UHy \end{bmatrix}$$

Fig.1b.

$$\underset{\underline{ff}}{\begin{bmatrix} FLHx \\ FLHy \\ FHLx \\ FHLy \end{bmatrix}} = \underset{S}{\begin{bmatrix} R & 0 & -R & 0 \\ 0 & 0 & 0 & 0 \\ -R & 0 & R & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \cdot \underset{\underline{uu}}{\begin{bmatrix} ULx \\ ULy \\ UHx \\ UHy \end{bmatrix}}$$

Fig.1c.

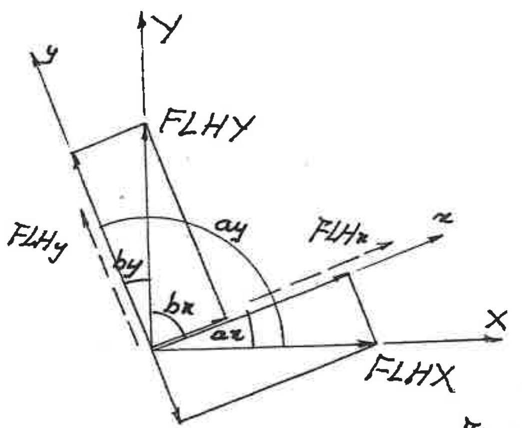


Fig.2.

$$\underset{T}{\begin{bmatrix} K & L & 0 & 0 \\ U & V & 0 & 0 \\ 0 & 0 & K & L \\ 0 & 0 & U & V \end{bmatrix}} \qquad \underset{T^{-1}}{\begin{bmatrix} K & \underline{U} & 0 & 0 \\ \underline{L} & V & 0 & 0 \\ 0 & 0 & K & \underline{U} \\ 0 & 0 & \underline{L} & V \end{bmatrix}}$$

K=Cos(ax) and L=Cos(bx),
ax from X to x and bx from Y to x.

U=Cos(ay) and V=Cos(by),
ay from X to y and by from Y to y.

2a. Space trusses.

2a.1. The member stiffness matrix of a member of a plane truss.

Derivation of the relation between member end forces, siffness matrix and member end displacements w.r.t. construction axis system X-Y. The member axis system is x-y.

Fig.1a.
The relation between member end forces and and member end displacements like found on page .

Fig.1b.
The member end forces perpendicular to the member axis x expressed in the member end displacements perpendicular to the member axis x. It concerns a truss member with member end forces FLHy and FHLy equal zero.

Fig.1c.
Both figures composed deliver the here drawn relation $\underline{ff} = S\ \underline{uu}$.

Fig.2.
The member end forces FLHx, FLHy, FHLx and FHLy w.r.t. the member axis system x-y can be expressed in the member end forces FLHX, FLHY, FHLX and FHLY w.r.t. the construction axis system X-Y. Below shown in matrix form in which T is the socalled transformation matrix.

$$\underset{\underline{ff}}{\begin{bmatrix} FLHx \\ FLHy \\ FHLx \\ FHLy \end{bmatrix}} = \underset{T}{\begin{bmatrix} Cos(ax) & Cos(bx) & 0 & 0 \\ Cos(ay) & Cos(by) & 0 & 0 \\ 0 & 0 & Cos(ax) & Cos(bx) \\ 0 & 0 & Cos(ay) & Cos(by) \end{bmatrix}} \cdot \underset{\underline{f}}{\begin{bmatrix} FLHX \\ FLHY \\ FHLX \\ FHLY \end{bmatrix}}$$

In similar way follows the relation $\underline{uu} = T\ \underline{u}$.

$$\underset{\underline{uu}}{\begin{bmatrix} ULx \\ ULy \\ UHx \\ UHy \end{bmatrix}} = \underset{T}{\begin{bmatrix} Cos(ax) & Cos(bx) & 0 & 0 \\ Cos(ay) & Cos(by) & 0 & 0 \\ 0 & 0 & Cos(ax) & Cos(bx) \\ 0 & 0 & Cos(ay) & Cos(by) \end{bmatrix}} \cdot \underset{\underline{u}}{\begin{bmatrix} ULX \\ ULY \\ UHX \\ UHY \end{bmatrix}}$$

In $\underline{ff} = S\ \underline{uu}$ of Fig.1c with

$\underline{ff} = T\ \underline{f}$ and $\underline{uu} = T\ \underline{u}$ follows $\underline{T\ \underline{f} = S\ T\ \underline{u}}$.

$T\ \underline{f}$ and $S\ T\ \underline{u}$ multiplied by the inverse

T gives $T^{-1}\ T\ \underline{f} = T\ \ S\ T\ \underline{u}$.

T times T gives unity matrix I, so thet

$\underline{\underline{f} = T^{-1}\ S\ T\ \underline{u}}$ with $\underline{S5 = T^{-1}\ S\ T}$.

Next the matrix multiplications ate carried out, first
S times T and after that T times S T.

$$\begin{bmatrix} R & 0 & -R & 0 \\ 0 & 0 & 0 & 0 \\ -R & 0 & R & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} K & L & 0 & 0 \\ U & V & 0 & 0 \\ 0 & 0 & K & L \\ 0 & 0 & U & V \end{bmatrix} = \begin{bmatrix} R*K & R*L & -R*K & -R*L \\ 0 & 0 & 0 & 0 \\ -R*K & -R*L & R*K & R*L \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

S T (S T)

$$\begin{bmatrix} K & \underline{U} & 0 & 0 \\ \underline{L} & V & 0 & 0 \\ 0 & 0 & K & \underline{U} \\ 0 & 0 & \underline{L} & V \end{bmatrix} \cdot \begin{bmatrix} R*K & R*L & -R*K & -R*L \\ 0 & 0 & 0 & 0 \\ -R*K & -R*L & R*K & R*L \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} R*K\^2 & R*K*L & -R*K\^2 & -R*K*L \\ R*K*L & R*L\^2 & -R*K*L & -R*L\^2 \\ -R*K\^2 & -R*K*L & R*K\^2 & R*K*L \\ -R*K*L & -R*L\^2 & R*K*L & R*L\^2 \end{bmatrix}$$

T-1 (S*T) S5

Fig.3a.

After the two matrix multiplications the final member stiffness matrix S5 has arisen.
Fig.3a.
Another way. It is not necessary to apply the 4 x 4 matrices since FLHy=0 and FHLy=0.
Fig.3b.
One may $\underline{ff}$ = T $\underline{u}$ and $\underline{uu}$ = T $\underline{u}$ of the preceding page represent in a 'shorter' way by omitting FLHy, FHLy, ULy and UHy.
Again one may apply, preceding page,

$\underline{f}$ = T S T $\underline{u}$ with S5 = T S T so that

$\underline{f}$ = S5 $\underline{u}$.

$$\begin{bmatrix} FLHx \\ FHLx \end{bmatrix} = \begin{bmatrix} R & -R \\ -R & R \end{bmatrix} \cdot \begin{bmatrix} ULx \\ UHx \end{bmatrix} \quad 1)$$

$$\underset{\underline{ff}}{\begin{bmatrix} FLHx \\ FHLx \end{bmatrix}} = \underset{T}{\begin{bmatrix} K & L & 0 & 0 \\ 0 & 0 & K & L \end{bmatrix}} \cdot \underset{\underline{f}}{\begin{bmatrix} FLHX \\ FLHY \\ FHLX \\ FHLY \end{bmatrix}} \quad 2)$$

The mtrix multiplications can be carried out because the number of elements of a row of the first 1) is equal to the number of elements of a column of the second 2).
For example, element (2,3) of (S T) is $\underline{R*K}$, the 2nd row of S tmes the 3rd column of T, (-R)(0) + R*K= R*K.
Element (2,4) is R*L. The second row of S times the third column of T, (-R)(0) + R*L= R*L.

$$\underset{\underline{uu}}{\begin{bmatrix} U1x \\ Uhx \end{bmatrix}} = \underset{T}{\begin{bmatrix} K & L & 0 & 0 \\ 0 & 0 & K & L \end{bmatrix}} \cdot \underset{\underline{u}}{\begin{bmatrix} ULX \\ ULY \\ UHX \\ UHY \end{bmatrix}}$$

Next the inverse T-1 is multiplied by (S T).
Element (4,2) of S5 is -R*L^2, the fourth row of T-1 times the second column of (S T) is (0)(R*L) + (L)(-R*L)= -R*L^2. Etc.

Fig.3b.

$$\begin{bmatrix} R & -R \\ -R & R \end{bmatrix} \cdot \begin{bmatrix} K & L & 0 & 0 \\ 0 & 0 & K & L \end{bmatrix} = \begin{bmatrix} R*K & R*L & -R*K & -R*L \\ -R*K & -R*L & R*K & R*L \end{bmatrix}$$

S T (S T)

$$\begin{bmatrix} K & 0 \\ L & 0 \\ 0 & K \\ 0 & L \end{bmatrix} \cdot \begin{bmatrix} R*K & R*L & -R*K & -R*L \\ -R*K & -R*L & R*K & R*L \end{bmatrix} = \begin{bmatrix} R*K\^2 & R*K*L & -R*K\^2 & -R*K*L \\ R*K*L & R*L\^2 & -R*K*L & -R*L\^2 \\ -R*K\^2 & -R*K*L & R*K\^2 & R*K*L \\ -R*K*L & -R*L\^2 & R*K*L & R*L\^2 \end{bmatrix}$$

T-1 (S T) S5

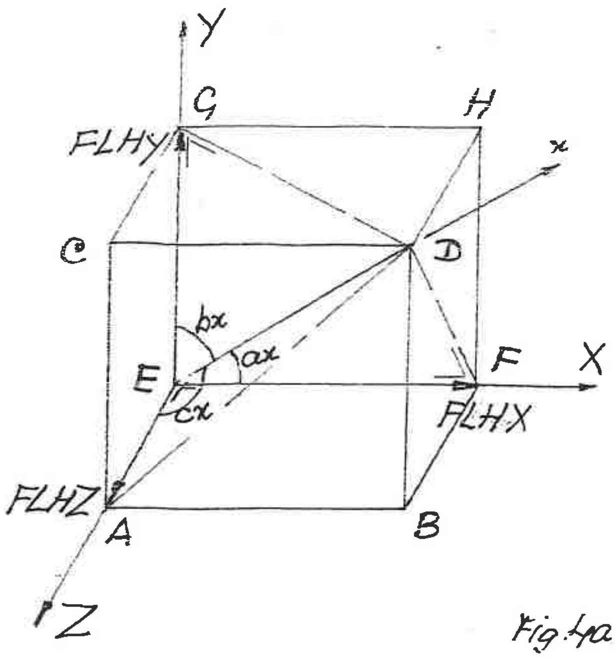


Fig.4a.

2a.2. The member stiffness matrix of a member of a space truss.

Fig. 4a en 4b.
Member end force FLHx consists of the components of the member end forces FLHX, FLHY and FLHZ w.r.t. the construction axis system X-Y-Z.

The forces FLHx and FHLx are directed like the x axis, the concerning shown angles are,
ax the angle between X- and x- axis,
bx the angle between Y- and x- axis, and
cx the angle between Z- and x- axis.

$$\begin{bmatrix} FLHx \\ FHLx \end{bmatrix} = \begin{bmatrix} R & -R \\ -R & R \end{bmatrix} \begin{bmatrix} Ulx \\ UHx \end{bmatrix} \qquad R=EA/L$$

See the three figures, fig.4b with the components. With the cosines of the shown angles, ax, bx and cx, follows for FLHx at member end L,
FLHx= FLHX*Cos(ax) +FLHY*Cos(bx) +FLHZ*Cos(cx).
And for FHLx at member end H,
FHLx= FHLX*Cos(ax) +FHLY*Cos(bx) +FHLZ*Cos(cx).

Force FLHy at member end L is perpendicular to the x axis. For an y axis perpendicular on the x axis, not shown in the figure, are
ay the angle between X- and y- axis,
by the angle between Y- and y- axis, and
cy the angle between Z- and y- axis.

Then follows for force FLHy at member end L the sum of components,
FLHy= FLHX*Cos(ay) +FLHY*Cos(by) +FLHZ*Cos(cy).
And for FHLy at member end H perpendicular to the x axis follows
FHLy= FHLX*Cos(ay) +FHLY*Cos(by) +FHLZ*Cos(cy).

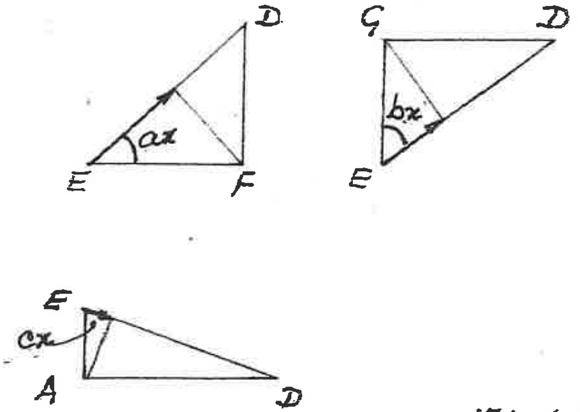


Fig.4b.

Force FLHz at member end L and FHLz at member end H are perpendicular to the x axis. For a z axis perpendicular to the x axis are, not shown in the figure,
az the angle between X- and z- axis,
bz the angle between Y- and z- axis, and
cz the angle between Z- and z- axis.

And follow like above the equations,
FLHz= FLHX*Cos(az) +FLHY*Cos(bz) +FLHZ*Cos(cz)
and, with HL i.s.o. LH,
FHLz= FHLX*Cos(az) +FHLY*Cos(bz) +FHLZ*Cos(cz).

The six (underlined) equations are given here below in matrix form ff = T f. The position of a member is determined by the x axis along the member. The x axis determines the y axis and thus the z axis, or determines the z axis and thus the y axis. A choice has to be made.

$$\underbrace{\begin{bmatrix} FLHx \\ FLHy \\ FLHz \\ FHLx \\ FHLy \\ FHLz \end{bmatrix}}_{ff} = \underbrace{\begin{bmatrix} Cos(ax) & Cos(bx) & Cos(cx) & 0 & 0 & 0 \\ Cos(ay) & Cos(by) & Cos(cy) & 0 & 0 & 0 \\ Cos(az) & Cos(bz) & Cos(cz) & 0 & 0 & 0 \\ 0 & 0 & 0 & Cos(ax) & Cos(bx) & Cos(cx) \\ 0 & 0 & 0 & Cos(ay) & Cos(by) & Cos(cy) \\ 0 & 0 & 0 & Cos(az) & Cos(bz) & Cos(cz) \end{bmatrix}}_{T} \cdot \underbrace{\begin{bmatrix} FLHX \\ FLHY \\ FLHZ \\ FHLX \\ FHLY \\ FHLZ \end{bmatrix}}_{f}$$

The elements of transformation matrix T of the preceding page can be simplyfied by replacing them by letters.

$$T = \begin{bmatrix} K & L & M & 0 & 0 & 0 \\ U & V & W & 0 & 0 & 0 \\ X & Y & Z & 0 & 0 & 0 \\ 0 & 0 & 0 & K & L & M \\ 0 & 0 & 0 & U & V & W \\ 0 & 0 & 0 & X & Y & Z \end{bmatrix}$$

T

K=Cos(ax)	L=Cos(bx)	M=Cos(cx)
U=Cos(ay)	V=Cos(by)	W=Cos(cy)
X-Cos(az)	Y=Cos(bz)	Z=Cos(cz)

Since it concerns a member of a truss, with joints regarded as hinges, there are no member end forces at the member ends perpendicular tomember axis x. Like on page <u>ff</u> = T <u>f</u> and <u>uu</u> = T <u>u</u> can be simplyfied..

Member stiffness matrix S5 can be found by matrix multiplicatiom.

Rem. The variables U, V and W, X, Y and Z do not appear in S5. Here below T-1 (S T)= S5.

$$\underset{\underline{ff}}{\begin{bmatrix} FLHx \\ FHLx \end{bmatrix}} = \underset{T}{\begin{bmatrix} K & L & M & 0 & 0 & 0 \\ 0 & 0 & 0 & K & L & M \end{bmatrix}} \cdot \begin{bmatrix} FLHX \\ FLHY \\ FLHZ \\ FHLX \\ FHLY \\ FHLZ \end{bmatrix}$$

$$\underset{\underline{uu}}{\begin{bmatrix} ULx \\ Uhx \end{bmatrix}} = \underset{T}{\begin{bmatrix} K & L & M & 0 & 0 & 0 \\ 0 & 0 & 0 & K & L & M \end{bmatrix}} \cdot \begin{bmatrix} ULX \\ ULY \\ ULZ \\ UHX \\ UHY \\ UHZ \end{bmatrix}$$

$$\underset{S}{\begin{bmatrix} R & -R \\ -R & R \end{bmatrix}} \cdot \underset{T}{\begin{bmatrix} K & L & M & 0 & 0 & 0 \\ 0 & 0 & 0 & K & L & M \end{bmatrix}} = \underset{(S\ T)}{\begin{bmatrix} R*K & R*L & R*M & -R*K & -R*L & -R*M \\ -R*K & -R*L & -R*M & R*K & R*L & R*M \end{bmatrix}}$$

$$\underset{T-1}{\begin{bmatrix} K & 0 \\ L & 0 \\ M & 0 \\ 0 & K \\ 0 & L \\ 0 & M \end{bmatrix}} \cdot \underset{(S\ T)}{\begin{bmatrix} R*K & R*L & R*M & -R*K & -R*L & -R*M \\ -R*K & -R*L & -R*M & R*K & R*L & R*M \end{bmatrix}} = \underset{S5}{\left[\begin{array}{ccc|ccc} R*K\hat{}2 & R*K*L & R*K*M & -R*K\hat{}2 & -R*K*L & -R*K*M \\ R*K*L & R*L\hat{}2 & R*L*M & -R*K*L & -R*L\hat{}2 & -R*L*M \\ R*K*M & R*L*M & R*M\hat{}2 & -R*K*M & -R*L*M & -R*M\hat{}2 \\ \hline -R*K\hat{}2 & -R*K*L & -R*K*M & R*K\hat{}2 & R*K*L & R*K*M \\ -R*K*L & -R*L\hat{}2 & R*L*M & R*K*L & R*L\hat{}2 & R*L*M \\ -R*K*M & -R*L*M & -R*M\hat{}2 & R*K*M & R*L*M & R*M\hat{}2 \end{array}\right]}$$

<u>S5(2,3)</u> is second row of T-1 times third column of (S T) is L*(R*M) + 0*(-R*M)= <u>R*L*M</u>.

<u>S5(3,5)</u> is third row of T-1 times fifth column of (S T) is M*(-R*L) + 0*(R*L)= <u>-R*L*M</u>.

<u>S5(6,6)</u> is sixth row of T-1 times sixth column of (S T) is 0*(-R*M) + M*(R*M)= <u>R*M^2</u>.

Modulus of elasticity E in kN/m^2,

strain stiffness EA in (kN/m^2)x(m^2) is in kN,

member length L1 in m,

member stiffness factor R= EA/L1 in kN/m.

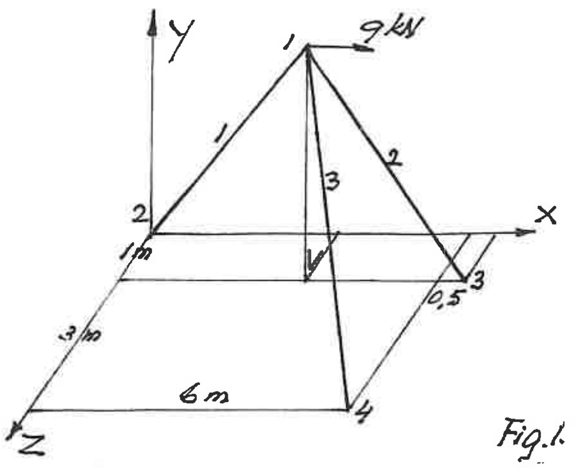


Fig.1.

Example.

Fig.1.
Three members and four joints. No own weight.Strain stiffness EA kN. (kN/m^2)*(m^2)
The coordinates of the joints.

X1(1)= 3,5   Y1(1)= 4,5   Z1(1)= 1,0 m
X1(2)= 0,0   Y1(2)= 0,0   Z1(2)= 0,0 m
X1(3)= 6,5   Y1(3)= 0,0   Z1(3)= 1,0 m
X1(4)= 6,0   Y1(4)= 0,0   Z1(4)= 4,0 m

Member 1. D1=X1(H)-X1(L)= 0-3,5= -3,5 m
D2=Y1(H)-Y1(L)= 0-4,5= -4,5 m
D3=Z1(H)-Z1(L)= 0-1,0= -1,0 m

L1=Sqr(D1^2+D2^2+D3^2)
=Sqr((-3,5)^2+(-4,5)^2+(-1,0)^2)
L1=Sqr(33,50)= 5,79 m

Member 1. K=D1/L1= -3,5/5,79= -0,60
L=D2/L1= -4,5/5,79= -0,78
M=D3/L1= -1,0/5,79= -0,17

A=R1*K^2=0,173*(-0,60)^2 = 0,062 EA
B=R1*K*L=0,173*(-0,60)(-0,78)= 0,081 EA
C=R1*K*M=0,173*(-0,60)(-0,17)= 0,018 EA

D=R1*L^2=0,173*(-0,78)^2 = 0,105 EA
E=R1*L*M=0,173*(-0,78)(-0,17)= 0,023 EA
F=R1*M^2=0,173*(-0,17)^2 = 0,005 EA

The member stiffness matrix S5 of member 1.
Siffness factor 'R=EA/L', R1=EA/5,79= 0,173.
The letters A, B, C, D, E and F represent the elements of matrix S5 of the preceding page. See the calculation on the left.

Here below S5 represented with letters.

$$\begin{bmatrix} FLHX \\ FLHY \\ FLHZ \\ FHLX \\ FHLY \\ FHLZ \end{bmatrix} = \left[\begin{array}{ccc|ccc} \mathbf{A} & \mathbf{B} & \mathbf{C} & -A & -B & -C \\ B & \mathbf{D} & \mathbf{E} & -B & -D & -E \\ C & E & \mathbf{F} & -C & -E & -F \\ \hline -A & -B & -C & A & B & C \\ -B & -D & -E & B & D & E \\ -C & -E & -F & C & E & F \end{array}\right] \cdot \begin{bmatrix} ULX \\ ULY \\ ULZ \\ UHX \\ UHY \\ UHZ \end{bmatrix}$$

Page 31 .

f		1	2	3	4	5	6	u
F12X	1	62	81	18	-62	-81	-18	UX1
F12Y	2	81	105	23	-81	-105	-23	UY1
F12Z	3	18	23	5	-18	-23	-5	UZ1
F21X	4	-62	-81	-18	62	81	18	UX2
F21Y	5	-81	-105	-23	81	105	23	UY2
F21Z	6	-18	-23	-5	28	23	5	UZ2

f = x EA/1000 S5 · u

An in similar way for member 2 and 3.
Member 2. D1=6,5-3,5= 3,0 D2= 0,0-4,5= -4,5
D3=1,0-1,0= 0,0 L2=Sqr(29,25)= 5,41 m
Member 3. D1=6,0-3,5= 2,5 D2= 0,0-4,5= -4,5
D3=4,0-1,0= 3,0 L3=Sqr(35,50)= 5,96 m

Member 2.

f		1	2	3	7	8	9
F13X	1	56	-84	0	-56	84	0
F13Y	2	-84	127	0	84	-127	0
F13Z	3	0	0	0	0	0	0
F31X	7	-56	84	0	56	-84	0
F31Y	8	84	-127	0	-84	127	0
F31Z	9	0	0	0	0	0	0

Member 3.

f		1	2	3	10	11	12
F14X	1	30	-54	35	-30	54	-35
F14Y	2	-54	97	-64	54	-97	64
F14Z	3	35	-64	42	-35	64	-42
F41X	10	-30	54	-35	30	-54	35
F41Y	11	54	-97	64	-54	97	-64
F41Z	12	-35	64	-42	35	-64	42

f = x EA/1000 S5

		1	2	3	4	5	6	7	8	9	10	11	12	
F12X +F13X +F14X	1	148	-57	53	-62	-81	-18	-56	84	0	-30	54	-35	9
F12Y +F13Y +F14Y	2	-57	329	-41	-81	-105	-23	84	-127	0	54	-97	64	0
F12Z +F13Z +F14Z	3	53	-41	47	-18	-23	-5	0	0	0	-35	64	-42	0
F21X	4	-62	-81	-18	62	81	18	.	.	.	.	.	.	0
F21Y	5	-81	-105	-23	81	105	23	.	.	.	.	.	.	0
F21Z	6	-18	-23	-5	28	23	5	.	.	.	.	.	.	0
F31X	7	-56	84	0	.	.	.	56	-84	0	.	.	.	0
F31Y	8	84	-127	0	.	.	.	-84	127	0	.	.	.	0
F31Z	9	0	0	0	.	.	.	0	0	0	.	.	.	0
F41X	10	-30	54	-35	.	.	.	.	.	.	30	-54	35	0
F41Y	11	54	-97	64	.	.	.	.	.	.	-54	97	-64	0
F41Z	12	-35	64	-42	.	.	.	.	.	.	35	-64	42	0

= x EA/1000 CC ·

1

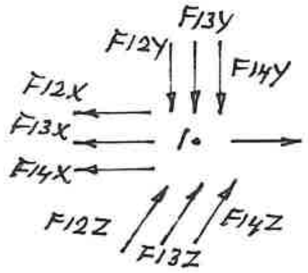


Fig.2.

Calculation of the member end forces with member ends 1.

Member 1.
F12X= 0,062EA(102,5/EA)
+0,081EA(3,8/EA)
+0,018EA(-112,3/EA)=
6,36+0,31-2,02= 4,65 kN

Further EA omitted.
F12Y= 0,081(102,5) +0,105(3,8)
+0,023(-112,3)=
8,30 +0,40 -2,58= 6,12 kN

F12Z= 0,018(102,5) +0,023(3,8)
0,005(-112,3)=
1,85 +0,09 -0,56= 1,38 kN

Member 2.
F13X= 0,056(102,5) -0,084(3,8) +0=
5,74 -0,32= 5,42 kN
F13Y= -8,13 kN en F13Z= 0,00 kN

Member 3.
F14X= 0,030(102,5) -0,054(3,8)
+0,035(-112,3)=
3,08 -0,21 -3,93=-1,06 kN
F14Y= 2,02 kN en F14Z=-1,37 kN.

Joint 1 in equilibrium?

Σ X =0 ? F12X+F13X+F14X-9,00=0 ?
4,65+5,42-1,06-9,00= 0,01 kN yes

Σ Y =0 ? F12Y+F13Y+F14Y=0 ?
6,12 -8,13 +2,02= 0,01 kN yes

Σ Z =0 ? F12Z+F13Z+F14Z=0 ?
1,38 +0 -1,37=0,01 kN yes

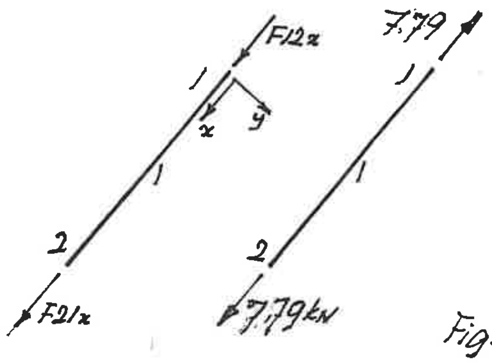


Fig.3.

F21X=-4,65 F21Y=-6,12 F21Z=-1,38 kN
F21x= K*F21X+L*F21Y+M*F21Z= 7,81 kN

The displacements of the three supports 2, 3 and 4 are prescribed, all zero,
UX2=0, UY2=0 and UZ2=0, UX3=0, UY3=0 and UZ3=0,
UX4=0, UY4=0 and UZ4=0.

$$
\begin{array}{cccccccccccc}1&2&3&4&5&6&7&8&9&10&11&12\end{array}
$$

$$
\begin{bmatrix}
148 & -57 & 53 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-57 & 329 & -41 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
53 & -41 & 47 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix} \underline{9}\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}
\begin{array}{c} UX1\\UY1\\UZ1\\UX2\\UY2\\UZ2\\UX3\\UY3\\UZ3\\UX4\\UY4\\UZ4 \end{array}
$$

x EA/1000 CC

The displacements UX1, UY1 and UZ1 of joint 1 are unknown. There are three equations left to solve.

EA(0,148*UX1 -0,057*UY1 +0,053*UZ1) = 9
EA(-0,057*UX1 0,329*UY1 -0,041*UZ1) = 0
EA(0,053*UX1 -0,041*UY1 +0,047*UZ1) = 0

Computer-GAUSS delivers
UX1=102,5/EA, UY1= 3,8/EA and UZ1=-112,3/EA.

Fig.2.
The member end forces are directed as assumed for the assumed X-, Y- and Z-axis. On the joints act these forces opposite directed, shown in the figure.
They are calculated with help of the member matrices S5, preceding page, f = S5 u.

Fig.3.
Assumed direction of the x axis from L to H.
The on the member ends acting member end forces FLHx and FHLx are directed like the x axis.
Member 1.
Calculation of member force F12x with L=1 and H=2. See the relation ff = T f of page 31 .
K=Cos(ax)= D1/L1=-3,5/5,79= -0,60
L=Cos(bx)= D2/L1=-4,5/5,79= -0,78
M=Cos(cx)= D3/L1=-1,0/5,79= -0,17

Then can be written for F12x, page ,
F12x= Cos(ax)*FLHX +Cos(bx)*FLHY +Cos(cx)*FLHZ
or
F12x= K*F12X + L*F12Y + M*F12Z
zodat
F12x= -0,60(4,65) +(-0,78)(6,12) +(-0,17)(1,38)
= -2,79 -4,77 -0,23= -7,79 kN

A hegative answer, in reality the force is not as assumed like the x axis directed, but opposite directed. The force pulls at the member end, the member is a tension member. For the other member end one will find F21x= 7,79 kN, the force pulls at member end 2.

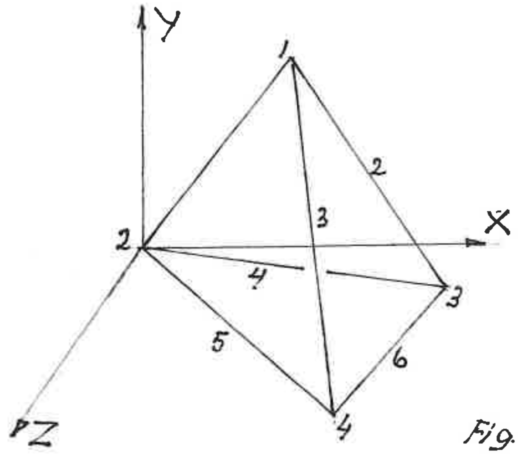


Fig.4

Fig.4.
Three members are added. The displacements of the supports 2 and 3 are prescribed and all zero, UX2=0, UY2=0, UZ2=0, UX3=0, UY3=0, UZ3=0.

The vertical displacement of of support 4 is prescribed, UY4=0. Joint/support 4 can displace horizontally according X and Z axis, UX4 and UZ4, being unknown. Then a free deformation due to temperature change is possible.

The three added members 4, 5 and 6 give three member stiffness matrices S5 shown on the left. They are put in construction matrix CC of page 32 . See here below.

Now 5 equations have to be solved to get the unknowns UX1, UY1, UZ1, UX4, and UZ4, schematically shown here below.

UX1	UY1	UZ1	UX4	UZ4	
0,148	-0,057	0,053	-0,030	-0,035	= 9
-0,057	0,329	-0,041	0,054	0,064	= 0
0,053	-0,041	0,049	-0,035	-0,042	= 0
-0,030	0,054	-0,035	0,134	0,046	= 0
-0,035	0,064	-0,042	0,046	0,406	= 0

Computer-GAUSS delivers

UX1=104,1/EA, UY1=4,9/EA, UZ1=-123,5/EA,

UX4=-9,7/EA and UZ4=-3,5/EA.

Calculation of the member end forces F12Z, F13Z and F14Z directed like Z, on joint 1 opposite directed. See the concerning member matrices. EA is omitted.

F12Z= 0,018(104,1)+0,023(4,9)+0,005(-123,5)=
= 1,87 + 0,11 -0,62 = 1,36 kN

F13Z= 0 kN (Member 2 in the vertical plane.)

F14Z= 0,035(104,1)-0,064(4,9)+0,042(-123,5)
-0,035(-9,7)-0,042(-3,5)

= 3,64 -0,31 -5,19 +0,34 +0,15 =-1,37 kN

Σ Z =0 ? 1,37 +0 -1,37= 0 yes

Member 4.

		4	5	6	7	8	9
F23X	4	149	0	23	-149	0	-23
F23Y	5	0	0	0	0	0	0
F23Z	6	23	0	3	-23	0	-3
=							
F32X	7	-149	0	-23	149	0	23
F32Y	8	0	0	0	0	0	0
F32Z	9	-23	0	-3	23	0	3

Member 5.

		4	5	6	10	11	12
F24X	4	96	0	63	-96	0	-63
F24Y	5	0	0	0	0	0	0
F24Z	6	63	0	42	-63	0	-42
=							
F42X	10	-96	0	-63	96	0	63
F42Y	11	0	0	0	0	0	0
F42Z	12	-63	0	-42	63	0	42

Member 6.

		7	8	9	10	11	12
F34X	7	8	0	-52	-8	0	52
F34Y	8	0	0	0	0	0	0
F34Z	9	-52	0	322	52	0	-322
=							
F43X	10	-8	0	52	8	0	-52
F43Y	11	0	0	0	0	0	0
F43Z	12	52	0	-322	-52	0	322

In these S5's no combinations 1-2-3 and therefore no alterations of the earlier CC of page .

		1	2	3	4	5	6	7	8	9	10	11	12		
F12X +F13X +F14X	1	148	-57	53	-62	-81	-18	-56	84	0	-30	54	-35		9
F12Y +F13Y +F14Y	2	-57	329	-41	-81	-105	-23	84	-127	0	54	-97	64		0
F12Z +F13Z +F14Z	3	53	-41	47	-18	-23	-5	0	0	0	-35	64	-42		0
F21X +F23X +F24X	4	-62	-81	-18	307	81	104	-149	0	-23	-96	0	-63		0
F21Y +F23Y +F24Y	5	-81	-105	-23	81	105	23	0	0	0	0	0	0		0
F21Z +F23Z +F24Z	6	-18	-23	-5	104	23	50	-23	0	-3	-63	0	-42		0
=														·	
F31X +F32X +F34X	7	-56	84	0	-149	0	-23	213	-84	-29	-8	0	52		0
F31Y +F32Y +F34Y	8	84	-127	0	0	0	0	-84	127	0	0	0	0		0
F31Z +F32Z +F34Z	9	0	0	0	-23	0	-3	-29	0	319	52	0	-322		0
F41X +F42X +F43X	10	-30	54	-35	-96	0	-63	-8	0	52	134	-54	46		0
F41Y +F42Y +F43Y	11	54	-97	64	0	0	0	0	0	0	-54	97	-64		0
F41Z +F42Z +F43Z	12	-35	64	-42	-63	0	-42	52	0	-322	46	-64	406		0

x EA/1000 CC

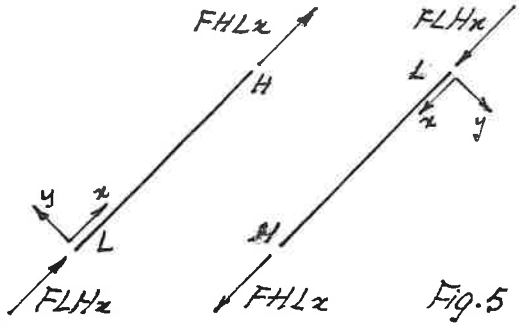


Fig.5

Member 3.
L=1 and H=4, coordinates page

D1=X1(4)-X1(1)= 6,0-3,5= 2,5 m
D2=Y1(4)-Y1(1)= 0,0-4,5= -4,5
D3=Z1(4)-Z1(1)= 4,0-1,0= 3,0
L3=Sqr(D1^2+D2^2+D3^2)= 5,96 m

K=Cos(ax)= D1/L3= -3,5/5,96= 0,42 rad
L=Cos(bx)= D2/L3= -4,5/5,96= -0,76
M=Cos(cx)= D3/L3= -1,0/5,96= -0,50

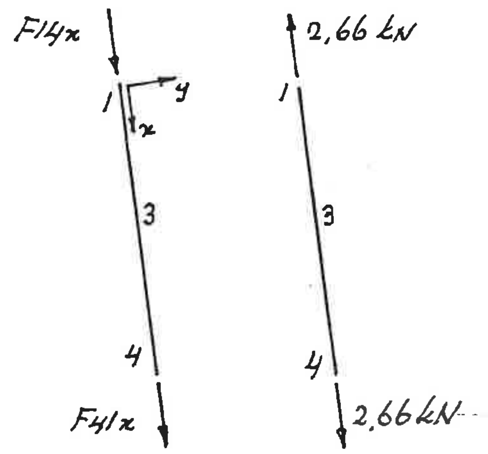


Fig.6.

Member 5.
L=2 and H=4.

D1=X1(4)-X1(2)= 6,0-0,0= 6,0 m
D2=Y1(4)-Y1(2)= 0,0-0,0= 0,0
D3=Z1(4)-Z1(2)= 4,0-0,0= 4,0
L5=Sqr(6,0^2+0,0^2+4,0^2)= 7,21 m

K=Cos(ax)= D1/L5= 6,0/7,21= 0,83 rad
L=Cos(bx)= D2/L5= 0,0/7,21= 0,00
M=Cos(cx)= D3/L5= 4,0/7,21= 0,56

Member 6.
L=3 and H=4.

D1=X1(4)-X1(3)= 6,0-6,5= -0,5 m
D2=Y1(4)-Y1(3)= 0,0-0,0= 0,0
D3=Z1(4)-Z1(3)= 4,0-1,0= 3,0
L5=Sqr(6,0^2+0,0^2+4,0^2)= 7,21 m

K=Cos(ax)= D1/L5=-0,5/7,21= -0,07 rad
L=Cos(bx)= D2/L5= 0,0/7,21= 0,00
M=Cos(cx)= D3/L5= 3,0/7,21= 0,42

Fig.5.
The member end forces FLHx and FHLx with
L the lowest member end number and
H the highest member end number.

The x-y axes system assumed at L and the x axis directed from L to H. Same direction for the on the member ends acting member end forces FLHx and FLHy.

Assumed FLHx presses on member end L, the member is a compression member. A negative answer, then FLHx does not press as assumed but pulls at member end L, in that case is the member a tension member.
Assumed FHLx pulls at member end H, the member is a tension member. A negative answer, then FHLx does not pull at but presses on member end H and is the member a compression member.

Fig.6.
Calculation of member force F41x of member 3 with help of the member end forces F41X, F41Y and F41z. See S5 of member 3 page 32.

F41X=-0,030(104,1)+0,054(4,9)-0,035(-123,5)
+0,030(-9,7) -0,054(0) +0,035(-3,5)=
-3,12 +0,26 +4,32 -0,29 -0,12= 1,05 kN

F41Y= 0,054(104,1) -0,097(4,9) +0,064(-123,5)
-0,054(-9,7) +0,097(0) -0,064(-3,5)=
5,62 -0,47 -7,90 -0,52 +0,22= -2,01 kN

F41Z=-0,035(104,1) +0,064(4,9) -0,042(-123,5)
+0,035(-9,7) -0,064(0) +0,042(-3,5)=
-3,64 +0,31 +5,19 -0,34 -0,15= 1,37 kN

F41x= K*F41X +L*F41Y +M*F41Z=
= 0,42(1,05)+(-0,76)(-2,01)+(0,50)(1,37)
= 0,44 +1,53 +0,69 = 2,66 kN

A positive answer, F41x pulls at member end 4 as assumed.
Member 3 is a tension member, 2,66 kN.

If one calculates F14X, F14Y and F14Z for member end 1 of member 3, and next F14x then one will find F14x= -2,66 kN. A negative answer, F14x does not press on member end 1 as assumed but pulls at member end 1.
Member 3 is a tension member, 2,66 kN.

Calculation of member force F42x of member 5.
See S5 of member 5 the preceding page. EA and zero multiplications omitted.

F42X= 0,096(-9,7) +0,063(-3,5)=
-0,93 -0,22= -1,15 kN

F42Y= 0,00 kN

F42Z= 0,063(-9,7) +0,042(-3,5)=
-0,61 -0,15= -0,76 kN

F42x= K*F42X +L*F42Y +M*F42Z=
= 0,83(-1,15)+0(0)+0,56(-0,76)=
=-0,95 +0 -0,43= -1,38 kN

A negative answer, F42x does not pull at member end 4 as assumed but presses on member end 4.
Member 5 is a compression member, 1,38 kN. Etc.

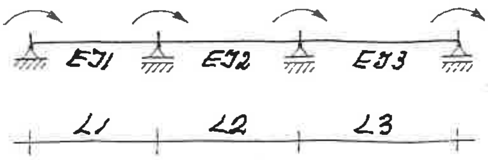


Fig. 1.

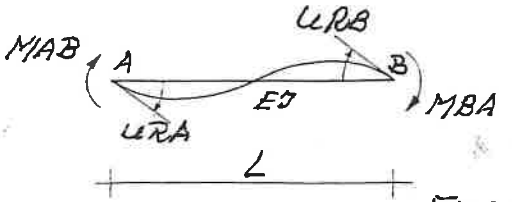


Fig. 2a.

=

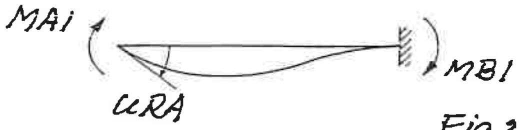


Fig. 2b.

+

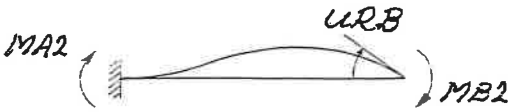


Fig. 2c

URA=MA1*L/(4*EI)   MA1=(4*EI/L)*URA

MB1=(2*EI/L)*URA

URB=MA2*L/(4*EI)   MA2=(2*EI/L)*URB

MB2=(4*EI/L)*URB

$$\begin{bmatrix} MAB \\ MBA \end{bmatrix} = \begin{bmatrix} 4EI/L & 2EI/L \\ 2EI/L & 4EI/L \end{bmatrix} \cdot \begin{bmatrix} URA \\ URB \end{bmatrix}$$

$$\underset{\underline{f}}{\begin{bmatrix} MAB \\ MBA \end{bmatrix}} = \underset{S5}{\begin{bmatrix} D & E \\ E & D \end{bmatrix}} \cdot \underset{\underline{u}}{\begin{bmatrix} URA \\ URB \end{bmatrix}}$$

'forces' - forces or moments

'displacements' - rotations or displacements

3. Continuous beams over several supports **without** (vertical) support displacements (translations) **without** internal hinges between the supports. Beams/members.

Fig.1.
A continuous beam on four supports, the joints at the supports represented with the short line pieces. For now no joints betwee the supports.
The joints are loaded with joint load moments, assumed direction to the right.
The joints rotate by deformation of the members due to member loads and joint loads.

3.1. The relation between member end moments and joint rotations of a member on 2 supports.

Fig.2a.
The beam/member is drawn separated from the supports.
The member is drawn separated from the joints.
On the member ends act member end moments MAB and MBA, assumed direction to the right.
The member end rotations, slope deflections, URA and URB, assumed direction to the right.
This member with moments MAB and MBA, and rotations URA and URB, can be regarded as the sum of figure 2a and 2b.

Fig.2b.
The member is clamped on the right. To an assumed rotation URA to the right belongs a to the right acting member end moment MA1.
By deformation of the member arises a clamp moment to the right at B.
According to the formula given on page 97 is the slope deflection URA at due to MA1

URA= MA1*L/(4*EI), and is MB1=MA1/2 so that

<u>MA1= (**4***EI/L)*URA</u>   and

<u>MB1= (2*EI/L)*URA</u>.   (Rem. F=(EA/L)*ΔL)

(4*EI/L) and (2*EI/L) are the member stiffness factors, or beam stiffness factors.

Fig.2c.
In similar way with the clamp at the left beam end with moment MB2 to the right and uURB to the right, and clamp moment MA2= MB2/2.

<u>MA2= (**2***EI/L)*URB</u>   and

<u>MB2= (**4***EI/L)*URB</u>.

When summed follow for figure 2a

MAB= MA1+MA2   1) and   MBA=MB1+MB2   2)   or

MAB= (4*EI/L)*<u>URA</u> +(2*EI/L)*<u>URB</u>   1)

MBA= (2*EI/L)*<u>URA</u> +(4*EI/L)*<u>URB</u>   2).

The relation between member end moments MAB and MBA and member end rotations URA and URB are represented on the left in matrix form.
(Rem. Spoken in general, member end 'forces' and member end 'displacements'.)

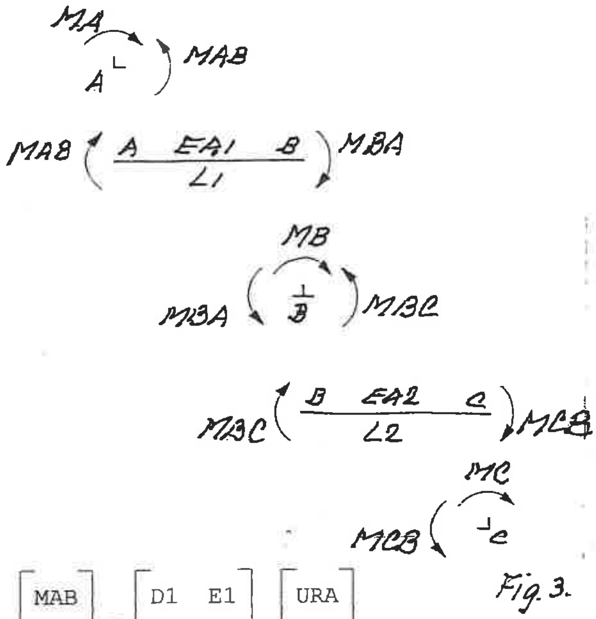


Fig. 3.

$$\begin{bmatrix} MAB \\ MBA \end{bmatrix} = \begin{bmatrix} D1 & E1 \\ E1 & \underline{D1} \end{bmatrix} \cdot \begin{bmatrix} URA \\ URB \end{bmatrix}$$

$$\underset{\underline{f}}{\begin{bmatrix} MBC \\ MCB \end{bmatrix}} = \underset{S5}{\begin{bmatrix} \underline{D2} & E2 \\ E2 & D2 \end{bmatrix}} \cdot \underset{\underline{u}}{\begin{bmatrix} URB \\ URC \end{bmatrix}}$$

$$\underset{\underline{f}}{\begin{bmatrix} MAB \\ MBA+MBC \\ MCB \end{bmatrix}} = \underset{CC}{\begin{bmatrix} D1 & E1 & 0 \\ E1 & \underline{D1+D2} & E2 \\ 0 & E2 & D2 \end{bmatrix}} \cdot \underset{\underline{u}}{\begin{bmatrix} URA \\ URB \\ URC \end{bmatrix}}$$

$$\underset{CC}{\begin{bmatrix} D1 & E1 & 0 \\ E1 & \underline{D1+D2} & E2 \\ 0 & E2 & D2 \end{bmatrix}} \cdot \underset{\underline{u}}{\begin{bmatrix} URA \\ URB \\ URC \end{bmatrix}} = \underset{\underline{f}}{\begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix}}$$

$$\underset{\substack{CC \\ \times\, EI/100}}{\begin{bmatrix} 1538 & 769 & 0 \\ 769 & 3516 & 928 \\ 0 & 964 & 1928 \end{bmatrix}} \cdot \underset{\underline{u}}{\begin{bmatrix} UR1 \\ UR2 \\ UR3 \end{bmatrix}} = \underset{\underline{f}}{\begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix}}$$

$$\underset{CC}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3516 & 0 \\ 0 & 82 & 164 \end{bmatrix}} \cdot \underset{\underline{u}}{\begin{bmatrix} UR1 \\ UR2 \\ UR3 \end{bmatrix}} = \underset{\underline{f}}{\begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix}}$$

Fig.3.
The consruction consisting of two beams of which beams and joints are separtated from each other.
The on the member ends acting member end moments are assumed directed to the right. On the the joints act these member end moments as large as but opposite directed, thus to the left. The joint rotations, or slope deflections, URA, URB and URC are assumed directed to the right.
Now there are two systems of equations shown on the left in matrix form, $\underline{f}$ = S5*$\underline{u}$ with the S5's as member stiffness matrices.

```
MAB= D1*URA +E1*URB       D1=(4*EI1/L1)
MBA= E1*URA +D1*URB       E1=(2*EI1/L1)

MBC= D2*URB +E2*URC       D2=(4*EI2/L2)
MCB= E2*URB +D2*URC       E2=(2*EI2/L2)
```

To compose to three equations, shown in matrix form, $\underline{f}$ = CC*$\underline{u}$ with CC as construction stiffness matrix.

```
MAB     = D1*URA +        E1*URB +  0*URC

MBA+MBC= E1*URA + (D1+D2)*URB + E2*URC

MCB     =  0*URA +        E2*URB + D2*URC
```

Joint load moments MA, MB and MC are assumed directed to the right. The elements of $\underline{f}$ of CC*$\underline{u}$ = $\underline{f}$ follow with equilibrium of the joints.

Σ mom. joint A =0	MAB-MA=0	MAB= MA
Σ mom. joint B =0	MBA+MBC-MB=0	MBA+MBC= MB
Σ mom. joint C =0	MCB-MC=0	MCB= MC

Fig.4.
UR2 is the unknown rotation. URA=0 and URC=0.
Support B with joint load moment MB= -5 kNm.

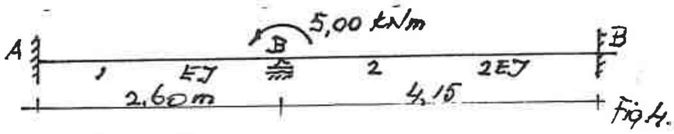


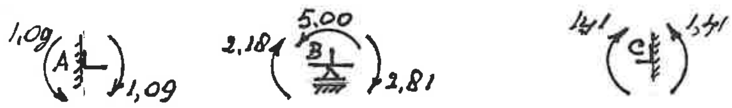
Fig.4.

Beam 1. L1=2,60 m
D1=4*1EI/2,60=1,538 E1=2*1EI/260=0,769 x EI
Beam 2. L2=4,15 m
D2=4*2EI/4,15=1,928 E2=2*2EI/4,15=0,964 x EI

With 3,516*UR2= -5 follows UR2= -1,42 rad.

MAB= E1*URB= 0,769(-1,42)= -1,09 kNm
MBA= D1*URB= 1,538(-1,42)= -2,18 kNm

MBC= D2*URB= 1,928(-1,42)= -2,81 kNm
MCB= E2*URB= 0,964(-1,42)= -1,41 kNm



The on the joints acting moments are drawn with their real directions with which follow the reaction moments at clamp A and C.

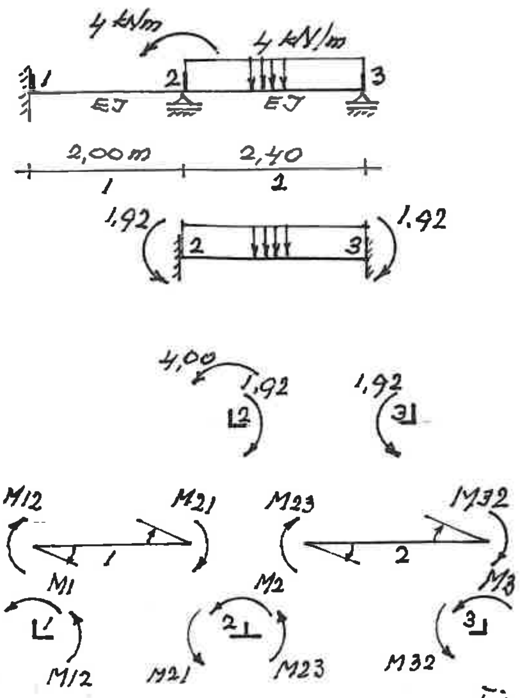


Fig.1.

$$\begin{bmatrix} M12 \\ M21 \end{bmatrix} = \begin{bmatrix} 200 & 100 \\ 100 & \underline{200} \end{bmatrix} \cdot \begin{bmatrix} UR1 \\ UR2 \end{bmatrix}$$

$$\begin{bmatrix} M23 \\ M32 \end{bmatrix} = \begin{bmatrix} \underline{164} & 82 \\ 82 & 164 \end{bmatrix} \cdot \begin{bmatrix} UR2 \\ UR3 \end{bmatrix}$$

$$\begin{bmatrix} 200 & 100 & 0 \\ 100 & 364 & 82 \\ 0 & 82 & 164 \end{bmatrix} \cdot \begin{bmatrix} UR1 \\ UR2 \\ UR3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2,08 \\ -1,92 \end{bmatrix}$$

x EI/100 CC $\underline{u}$ $\underline{f}$

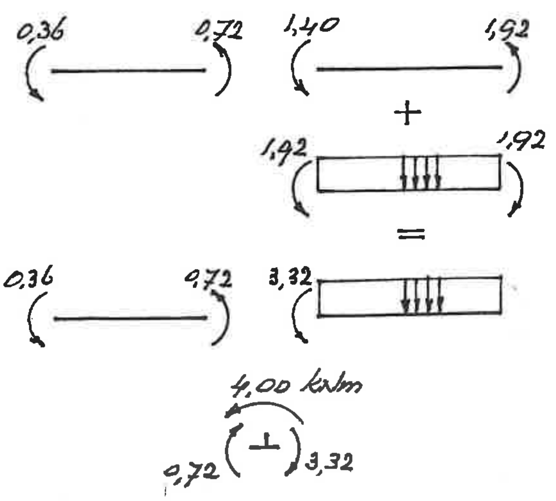


Fig.2.

Example.

Fig.1.
The beam consists of two parts with bending stiffness EI. Joint 2 is loaded with a joint load force of 4 kNm to the left, beam 2 loaded with a uniformly distributed load of 4 kN/m.

Beam end 3 of the second beam is regarded as a real joint with an unknown joint rotation UR3.

The joint load moments have given values M1=0, M2=-4 and M3=0 kNm. The joint load moments due to the beam loads, the primary moements are added.
Starting point are the joint rotations equal zero, the undeformed situation, so UR2=0 and UR3=0. For the at both ends clamped beam the beam end rotations are zero. The on the beam ends acting moments are
(1/12)*4*(2,4)^2= 1,92 kNm
with directions as drawn.
On the joints act moments as large as but opposite directed. On joint 2 to the right and on joint 3 to the left. With assumed direction for joint load moments to the right then follow
<u>M2=1,92-4,00= -2,08 kNm</u> and <u>M3= -1,92 kNm</u>.

The stiffness factors of the matrices S5.

Beam 1. L1=2,00 m
D1=4*EI/2,00= 2,00 E1=2*EI/2,00= 1,00 x EI

Beam 2. L2=2,40 m
D2=4*EI/2,40= 1,64 E2=1*EI/2,40= 1,20 x EI

On the left the beam end moments M12, M21 and M23 are given in matrix form $\underline{f}$ = S5*$\underline{u}$, next composed to $\underline{f}$ = CC*$\underline{u}$, a system of 3 equations.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 364 & 82 \\ 0 & 82 & 164 \end{bmatrix} \cdot \begin{bmatrix} UR1 \\ UR2 \\ UR3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2,08 \\ -1,92 \end{bmatrix}$$

x EI/100 CC $\underline{u}$ $\underline{f}$

Since UR1=0 two equations remain.

3,64*UR2 +0,82*UR3 = -2,08

0,82*UR2 +1,64*UR3 = -1,92 with which follow

<u>UR2= -0,36 rad</u> and <u>UR3=-0,99 rad</u> <u>/EI</u>

Fig.2.
M12= 1,00(-0,36)= -0,36 kNm
M21= 2,00(-0,36)= -0,72 kNm

M23= 1,64(-0,36) +0,82(-0,99)
=-0,59-0,81=-1,40 kNm

M32=0,82(-0,36)+1,64(-0,99)
=-0,30-1,62=-1,92 kNm

The beam end moments are drawn with their real directions.

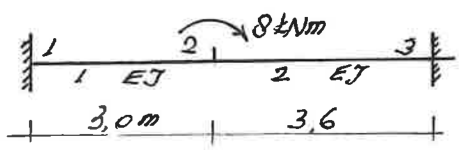


Fig.1.

$$\begin{bmatrix} M12 \\ M21 \end{bmatrix} = \begin{bmatrix} 1333 & 667 \\ 667 & 1333 \end{bmatrix} \cdot \begin{bmatrix} UR1 \\ UR2 \end{bmatrix}$$

$$\begin{bmatrix} M23 \\ M32 \end{bmatrix} = \begin{bmatrix} 1111 & 556 \\ 556 & 1111 \end{bmatrix} \cdot \begin{bmatrix} UR2 \\ UR3 \end{bmatrix}$$

x EI/1000

$$\begin{bmatrix} 1333 & 667 & 0 \\ 667 & 2444{+}1200 & 556 \\ 0 & 566 & 1111 \end{bmatrix} \cdot \begin{bmatrix} UR1 \\ UR2 \\ UR3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix}$$

x EI/1000 CC u f

spring constant MS2= 1,2EI kNm/rad

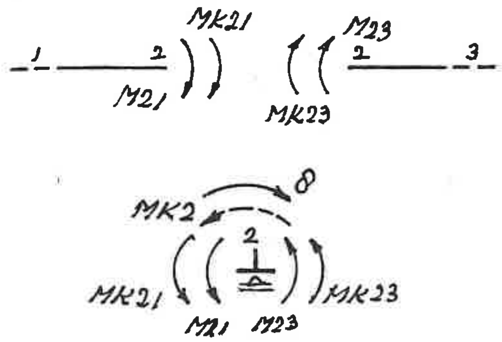


Fig.2.

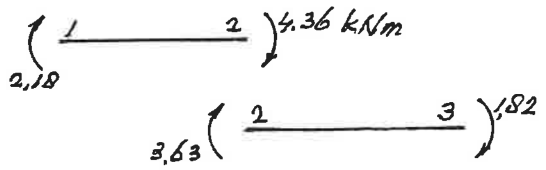


Fig.4a

Member end moments without clamp spring at joint 2.

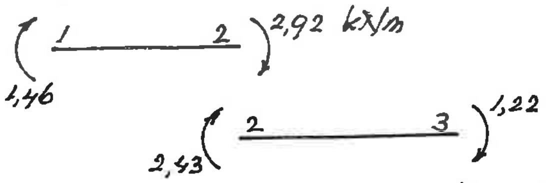


Fig.4b.

Member end moments with clamp spring at joint 2. MK2 is spring constant MS2 times angle UR2,

MK2=1,2EI*2,19/EI= 2,63 ≈ 2,64 kNm OK!!

The rotation spring and spring moment.

Fig.1.
Joint 2 with a joint load moment of 8 kNm to the right.

S51 with D1= 4EI/L= 4EI/3,0= 1,333 x EI
E1= 2EI/L= 2EI/3,0= 0,667 x EI

S52 with D2= 4EI/3,6= 1,111 x EI
E2= 2EI/3,6= 0,556 x EI

S51 and S52 form construction matrix CC.
With result like on the preceding page,

2,444EI*UR2= 8 so that UR2= 3,27/EI rad.

M12= 0,667EI*3,27/EI= 2,18 kNm
M21= 1,333EI*3,27/EI= 4,36 kNm

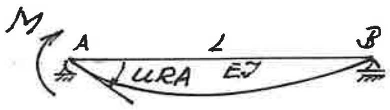
M23= 1,111(3,27)= 3,63 kNm and
M32= 0,556(3,27)= 1,82 kNm.

The rotation spring at joint 2.

Fig.2.
The spring will alter the beam end moments at joint 2 with MK21 and MK23 assume to the right. On the joint as large as but opposite directed thus to the left. Together the roatation spring moment to thr left MK2= MK21 + MK23.

MK2= MS2 * UR2, spring constant MS2 in kNm/rad.

Fig.3.



For this beam is URA= (M*L)/(3EI) or
M= (3EI/L) * URA with 'spring'constant (3EI/L) with which an idea of magnitude is given.

Σ mom. joint 2= 0 or M21+M23+MK1 -8= 0 or

0,667*UR1 +1,333*UR2 +1,111*UR2 +0,556*UR3

\+ MS1*UR2= 8

Suppose MS1= 1,2EI then follows (EI omitted)

0,667*UR1+(1,333+1,111+1,200)*UR2+0,556*UR3= 8.

One equation remains.
3,644EI*UR2 =8 so that UR2= 2,19/EI.

Fig.4a en 4b.
Without and with rotation spring.

Without	M21= 4,36	M23= 3,63	kNm	and
with	M21= 2,92	M23= 2,43	kNm	
Differences	MK21= 1,44	MK23= 1,20	kNm	

Spring moment

MK2= MK21+MK23= 1,44+1,20= 2,64 kNm

Σ mom. joint 2=0 MK2 +M21+M23 -8,00= 0 ?
2,64 +2,92 +2,43 -8,00 = -0,01 is OK.

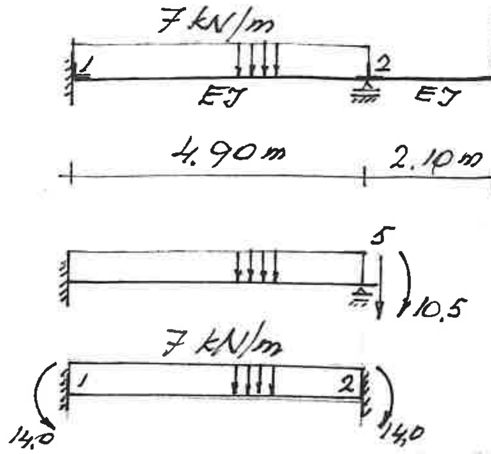


Fig.1.

$$\begin{bmatrix} M12 \\ M21 \end{bmatrix} = \begin{bmatrix} 816 & 408 \\ 408 & 816 \end{bmatrix} \cdot \begin{bmatrix} UR1 \\ UR2 \end{bmatrix}$$

$\underline{f}$ S5 $\underline{u}$

x EI/1000

M12= 0,816*UR1 +0,408*UR2

M21= 0,408*UR1 +0,816*UR2

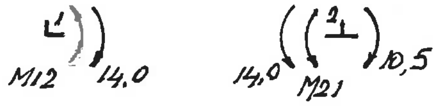


Fig.2.

$$\begin{bmatrix} 816 & 408 \\ 408 & 816 \end{bmatrix} \cdot \begin{bmatrix} UR1 \\ UR2 \end{bmatrix} = \begin{bmatrix} 14,0 \\ -3,5 \end{bmatrix}$$

S5 $\underline{u}$ $\underline{f}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 816 \end{bmatrix} \cdot \begin{bmatrix} UR1 \\ UR2 \end{bmatrix} = \begin{bmatrix} 0 \\ -3,5 \end{bmatrix}$$

x EI/1000

Fig.3a

+

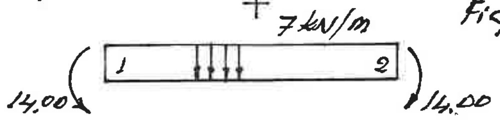


Fig.3b.

=

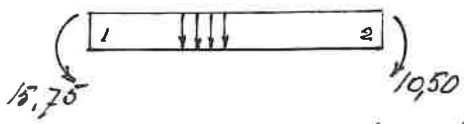


Fig.3c

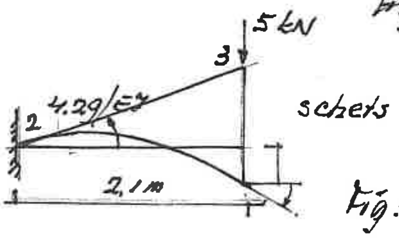


Fig.4.

Example.

Fig.1.
The beam with overhanging part is simplified to a single beam. The load at joint 2 due to the concentrated load force of 5 kN is found by re-solving 5 kN into a force of 5 kN at joint 2 plus a coupleof forces with a moment of 5*2,1= 10,5 kNm to the right, it is the joint load moment of joint 2.

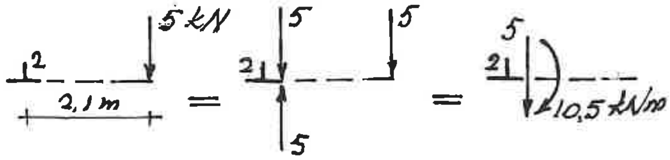


Fig.2.
The joint load moments due to the uniformly distributed load of 7 kN/m are (1/12)*7*(4,90)^2= 14,0 kNm, on joint 1 to the right, on joint 2 to the left.

The elements of member stiffness matrix S5.
D= 4EI/L= 4EI/4,90= 0,816 x EI
E= 2EI/L= 2EI/4,90= 0,408 x EI

On the left represented in matrix form
$\underline{f}$ = S5 * $\underline{u}$.

The elements of $\underline{f}$ in S5 * $\underline{u}$ = $\underline{f}$ follow with moment equilibrium of the joints.

Σ mom. joint 1=0
M12-14,0=0 M12= 14,0 kNm
Σ mom. joint 2=0
M21 +14,0 -10,5 =0 M21= -3,5 kNm

Since the rotation of joint 1 is known, UR1=0, is rotation UR2 the only unknown, follows
0,816EI*UR2= -3,5 ⇒ UR2= -4,29/EI rad

M12= 0,816EI*0 +0,408EI*(-4,29/EI)= -1,75 kNm
M21= 0,408EI*0 +0,816EI*(-4,29/EI)= -3,50 kNm

Fig.3a.
The member end moments due to the joint rotations UR1 and UR2 alone are drawn with their real directions.
Fig.3b.
The member end moments due to the load of 7 kN alone.
Fig.3c.
The sum of 3a and 3b gives the final member end moments.
Fig.4.
With 'forget-me-nots'. See page 97.
The slope deflection, rotation, as assumed to the right, the follows, see figure ,
UR3= 5*2,1^2/2EI-4,29/EI= 11,03/EI-4,29/EI= 6,74/EI, positive answer so as assumed to the right.

Suppose displacement UV3 downward then follows
UV3= 5*2,1^3/3EI-(4,29/EI)*2,10
= 15,44/EI-9,01/EI= 6,43/EI positive answer so as assumed downward.
See example, same construction page 45 .

4. Continuous beams over more than two supports **with** vertical joint displacements and **without** internal hinges between the supports.

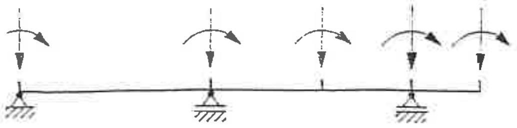


Fig.1.

Fig.1.
Not the supports alone seen as joints but also elsewhere places seen as joints.

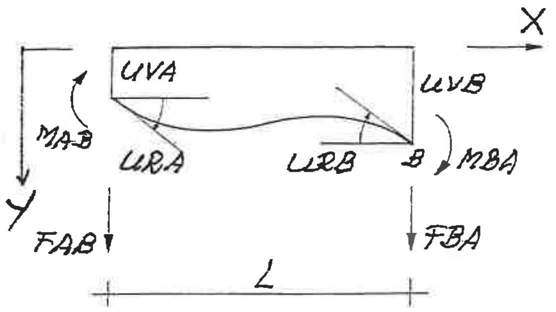


Fig.2a.

Fig.2a.
In the figure the assumptions for the joint rotations URA and URB are to the right, for the member end moments MAB and MBA to the right and for the vertical joint displacements UVA and UVB downward. Here UVB is drawn larger than UVA but one could have drawn UVA larger than UVB.

This drawn situation is the addition of four separated cases, fig.2a, 2b, 2c and 2d.

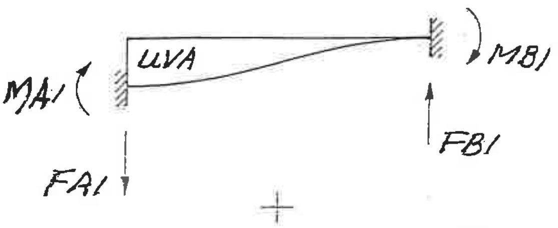


Fig.2b.

Fig.2b.
First alone the displacement UVA of joint A downward. At A and B no joint rotations. At A and B arise moments MA1 and MB1 to the right due to the deformation of the beam. Then at A and B have to arise reactions FA1 downward and FB1 upward to make equilibrium possible with the moments MA1 and MB1.
With the formulas of page 97 then follow

MA1= (6*EI/L^2)*UVA and MB1= (6*EI/L^2)*UVA,

FA1= (12*EI/L^3)*UVA and FB1= (12*EI/L^3)*UVA.

Next the influence of joint rotation URA of joint A to the right.

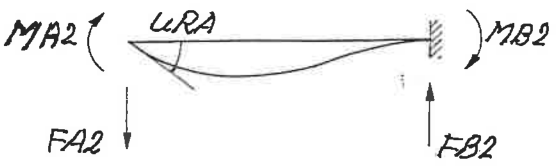


Fig.2c.

Fig.2c.
On page the beam end moments due to joint rotation URA to the right were found. Now named MA2 and MB2 instead of MA1 and MB1.

MA2= (**4***EI/L)*URA en MB2= (**2***EI/L)*URA.

Now also the beam end forces FA2 and FB2.

FA2= (6*EI/L^2)*URA and FB2= (6*EI/L^2)*URA.

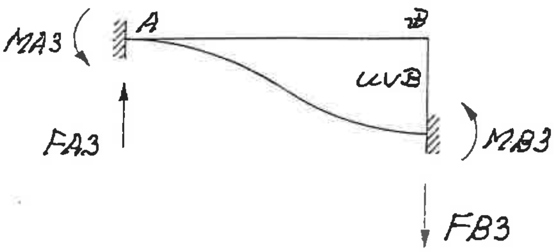


Fig.2d.

Fig.2d.
Alone the displacement of joint B over UVB downward. Joint A does not displace and there are no joint rotations. From the deformation follows the direction of the beam end moments and from them the two beam end forces which have to be in equilibrium with bot memnets.

Like with fig.2b follow with the formulas

MA3= (6*EI/L^2)*UVB and MB3= (6*EI/L^2)*UVB,

FA3= (12*EI/L^3)*UVB and FB3= (12*EI/L^3)*UVB.

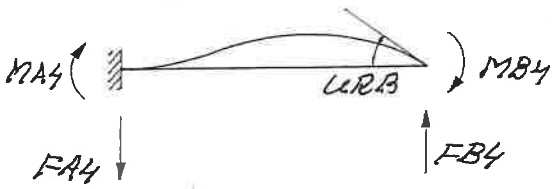


Fig.2e.

Fig.2e.

Like fig.2c but mirrored.

MA4= (**2***EI/L)*URB and MB4= (**4***EI/L)*URB.

Now also the beam end forces FA4 and FB4.

FA4= (6*EI/L^2)*URB and FB4= (6*EI/L^2)*URB.

$$\begin{bmatrix} FAB \\ MAB \\ FBA \\ MBA \end{bmatrix} = S5 \begin{bmatrix} UVA \\ URA \\ UVB \\ URB \end{bmatrix}$$

$\underline{f}$ S5 $\underline{u}$

$$\begin{bmatrix} 12EI/L\hat{}3 & 6EI/L\hat{}2 & -12EI/L\hat{}3 & 6EI/L\hat{}2 \\ 6EI/L\hat{}2 & 4EI/L & -6EI/L\hat{}2 & 2EI/L \\ -12EI/L\hat{}3 & -6EI/L\hat{}2 & 12EI/L\hat{}3 & -6EI/L\hat{}2 \\ 6EI/L\hat{}2 & 2EI/L & -6EI/L\hat{}2 & 4EI/L \end{bmatrix}$$

S5

$$\begin{bmatrix} A & B & -A & B \\ B & D & -B & E \\ -A & -B & A & -B \\ B & E & -B & D \end{bmatrix}$$

$$A= \frac{(kN/m\hat{}2)\,(m\hat{}4)}{(m\hat{}3)} = kN/m$$

$$B= \frac{(kN/m\hat{}2)\,(m\hat{}4)}{(m\hat{}2)} = kN$$

$$D= \frac{(kN/m\hat{}2)\,(m\hat{}4)}{m} = kNm$$

$$E= \frac{(kN/m\hat{}2)\,(m\hat{}4)}{m} = kNm$$

$$\begin{bmatrix} kN \\ kNm \\ kN \\ kNm \end{bmatrix} = \begin{bmatrix} kN/m & kN & kN/m & kN \\ kN & kNm & kN & kNm \\ kN/m & kN & kN/m & kN \\ kN & kNm & kN & kNm \end{bmatrix} \cdot \begin{bmatrix} m \\ 1 \\ m \\ 1 \end{bmatrix}$$

$\underline{f}$ S5 $\underline{u}$

In $\underline{u}$ a 1 for radians, no dimension.

Second row of S5 times column $\underline{u}$,

kNm = kN*m kNm*1 kN*m kNm*1

The final member end moments MAB and MBA, and member end forces FAB and FBA of figure 2a consist of the addition of moments and forces of the figures 2b, 2c, 2d and 2e.

On page 36 the force vector of a member consisted of MAB and MBA.
See $\underline{f}$ = S5*$\underline{u}$ given on the left. (The order FAB, MAB, FBA, MBA could gave been MAB, FAB, MBA, FBA if just consistent applied.)

<u>Determination of the elements of member stiffnes matrix S5.</u>

FAB see figure 2a is assumed downward, then FAB equals the sum of the forces downward minus the sum of the forces upward.

FAB= FA1 +FA2 +FA4 -FA3 and correct in order

<u>FAB= FA1 +FA2 -FA3 +FA4</u> 1)

= (12*EI/L^3)*UVA +(6*EI/L^2)*URA

-(12*EI/L^3)*UVB +(6*EI/L^2)*URB

MAB is assumed to the right, then MAB equals the sum of moments to the right minus the sum of moments to the left.

MAB= MA1 +MA2 +MA4 -MA3 and in correct order

<u>MAB= MA1 +MA2 -MA3 +MA4</u> 2)

= (6*EI/L^2)*UVA +(4*EI/L)*URA

-(6*EI/L^2)*UVB +(2*EI/L)*URB

Similar for force FBA and moment MBA at member end B.

FBA= FB3 -FB1 -FB2 -FB4 or

<u>FBA=-FB1 -FB2 +FB3 -FB4</u> 3)

=-(12*EI/L^3)*UVA -(6*EI/L^2)*URA

+(12*EI/L^3)*UVB -(6*EI/L^2)*URB

MBA= MB1 +MB2 +MB4 -MB3 or

<u>MBA= MB1 +MB2 -MB3 +MB4</u> 4)

= (6*EI/L^2)*UVA +(2*EI/L)*URA

-(6*EI/L^2)*UVB +(4*EI/L)*URB

On the left the concerning elements are placed in S5. One sees the symmetrie. Below the elements are replaced by letters A, B, D and E, a few with a minus sign. Only four values have to be calculated.
kend.

<u>A=12*EI/L^3 B=6*EI/L^2 D=4*EI/L E=2*EI/L</u>

On the left the dimensions of the elements of $\underline{f}$, S5 and $\underline{u}$ represented. An element of $\underline{f}$ equals a row of of S5 times a column $\underline{u}$.

Exchange of A and B.

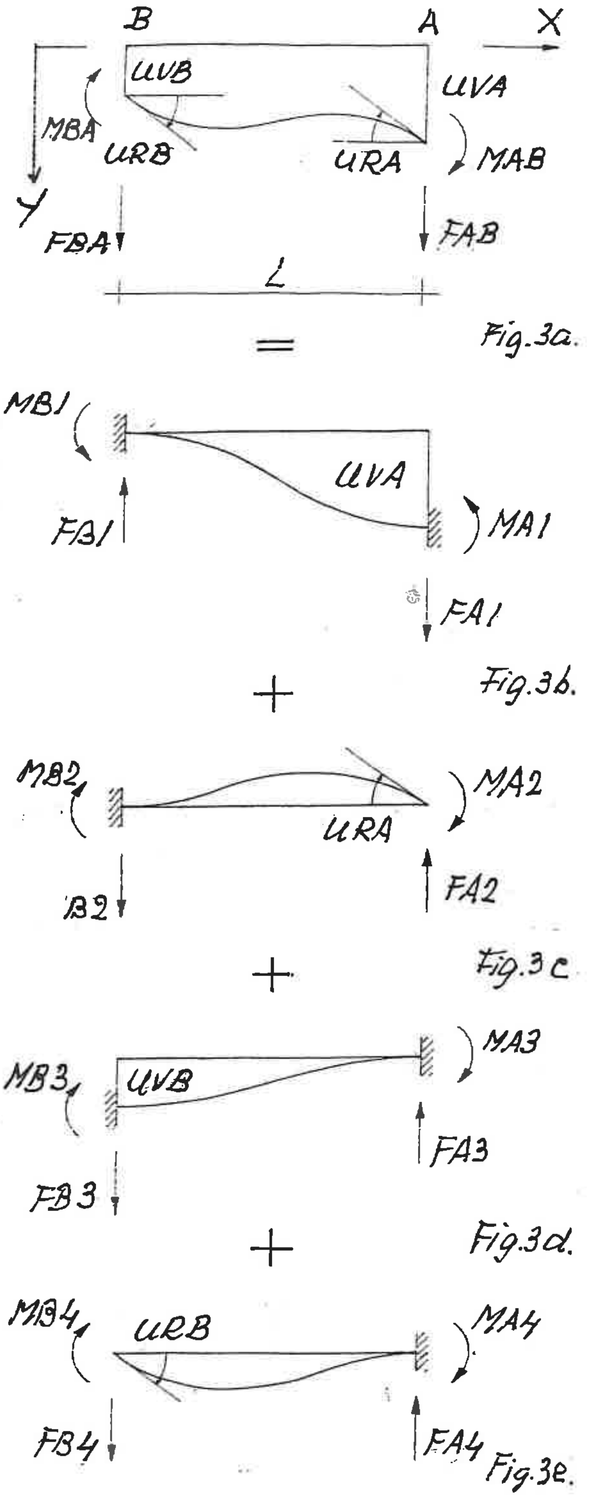


Fig.3a.
A on the right and B on the left. Assumptions like before, member end rotations YRA and URB to the right, the figure fits to them, and matching member end moments MAB and MBA to the right, and member end forces FAB and FBA downward.

Fig.3b.
To the displacement UVA alone belong the moments MA1 and MB1 to the left. The member end forces FA1 and FB1 are as large as but opposite directed and form a couple of forces with a moment to the right for equilibrium. With the formulas of page then follow

MA1= (6*EI/L^2)*UVA and MB1= (6*EI/L^2)*UVA,
FA1= (12*EI/L^3)*UVA and FB1= (12*EI/L^3)*UVA.

Fig.3c.
To a rotation URA belong the moments MA2 and MB2 to the right. The forces FA2 and FB2 form a couple to the left making equilibrium.

MA2= (**4***EI/L)*URA and MB2= (**2***EI/L)*URA,
FA2= (6*EI/L^2)*URA and FB2= (6*EI/L^2)*URA.

Fig.3d.
Like figure 3a but now only UVB with the moments MA3 and MB3 to the right and the forces FA3 and FB3 making a couple to the left.

MA3= (6*EI/L^2)*UVB and MB3= (6*EI/L^2)*UVB,
FA3= (12*EI/L^3)*UVB and FB3= (12*EI/L^3)*UVB.

Fig.3e.
Finally the assumed slope deflection/rotation URB to the right with matching moments MA4 and MB4 and couple forces FA4 and FB4 with moment to the left.

MA4= (**2***EI/L)*URB and MB4= (**4***EI/L)*URB,
FA4= (6*EI/L^2)*URB and FB4= (6*EI/L^2)*URB

FAB= FA1 -FA2 -FA3 -FA4

= (12*EI/L^3)*UVA -(6*EI/L^2)*URA
-(12*EI/L^3)*UVB -(6*EI/L^2)*URB   A-B -A-B

MAB=-MA1 +MA2 +MA3 +MA4

=-(6*EI/L^2)*UVA +(**4***EI/L)*URA
+(6*EI/L^2)*UVB +(**2***EI/L)*URB   -B D B E

FBA=-FB1 +FB2 +FB3 +FB4

=-(12*EI/L^3)*UVA +(6*EI/L^2)*URA
(12*EI/L^3)*UVB +(6*EI/L^2)*URB   -A B A B

MBA=-MB1 +MB2 +MB3 +MB4

=-(6*EI/L^2)*UVA +(**2***EI/L)*URA
+(6*EI/L^2)*UVB +(**4***EI/L)*URB   -B E B D

On the left both possible S5's are represented by the letters with earlier found values
A=12*EI/L^3   B=6*EI/L^2   D=4*EI/L   E=2*EI/L.

$$\left[\begin{array}{cc|cc} A & B & -A & B \\ B & D & -B & E \\ \hline -A & -B & A & -B \\ B & E & -B & D \end{array}\right] \qquad \left[\begin{array}{cc|cc} A & -B & -A & -B \\ -B & D & B & E \\ \hline -A & B & A & B \\ -B & E & B & D \end{array}\right]$$

A — B / L — H   B — A / H — L

L is the lowest member end number and H the highest member end number, L - A, H - B. See the diagonal symmetrie.

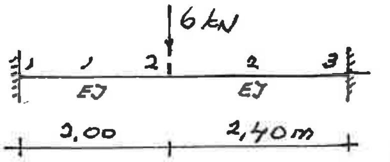


Fig.1.

$$\begin{bmatrix} F12 \\ M12 \\ F21 \\ M21 \end{bmatrix} = \begin{bmatrix} 150 & 150 & -150 & 150 \\ 150 & 200 & -150 & 100 \\ -150 & -150 & \underline{150} & \underline{-150} \\ 150 & 100 & \underline{-150} & \underline{200} \end{bmatrix} \cdot \begin{bmatrix} UV1 \\ UR1 \\ \underline{UV2} \\ \underline{UR2} \end{bmatrix}$$

x EI/100 S51

$$\begin{bmatrix} F23 \\ M23 \\ F32 \\ M32 \end{bmatrix} = \begin{bmatrix} \underline{87} & \underline{104} & -87 & 104 \\ \underline{104} & \underline{167} & -104 & 84 \\ -87 & -104 & 87 & -104 \\ 104 & 84 & -104 & 167 \end{bmatrix} \cdot \begin{bmatrix} \underline{UV2} \\ \underline{UR2} \\ UV3 \\ UR3 \end{bmatrix}$$

$$\begin{bmatrix} 150 & 150 & -150 & 150 & 0 & 0 \\ 150 & 200 & -150 & 84 & 0 & 0 \\ -150 & -150 & \underline{237} & \underline{-46} & -87 & 104 \\ 150 & 100 & \underline{-46} & \underline{367} & -104 & 84 \\ 0 & 0 & -87 & -104 & 87 & -104 \\ 0 & 0 & 104 & 84 & -104 & 167 \end{bmatrix} \cdot \begin{bmatrix} UV1 \\ UR1 \\ \underline{UV2} \\ \underline{UR2} \\ UV3 \\ UR3 \end{bmatrix}$$

CC

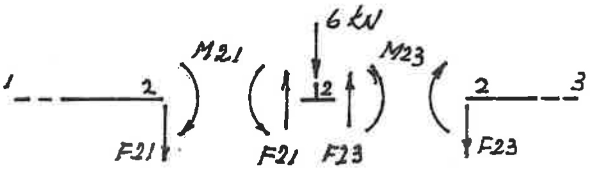


Fig.2.

Member 1.

F12= EI(-1,50(2,59/EI)+1,50(0,33/EI))
= -3,89 +0,50 = -3,39 kN

M12= -1,50(2,59) +1,00(0,33)
= -2,89 +0,33 = -3,56 kNm

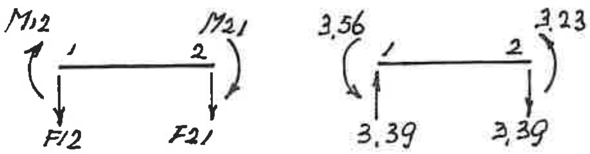


Fig.3.

F21= 1,50(2,59) -1,50(0,33)
= 3,89 -0,50 = 3,39 kN

M21= -1,50(2,59) +2,00(0,33)
= -3,89 +0,66 = -3,23 kNm

Example.

Fig.1.
At the load force of 6 kN a joint 2 is assumed. Then there are tow beams/members with lengths 2,00 m and 2,40 m. Joint 2 can displace horizontally.

Each member stiffness matrix S5 has 4 x 4 elements with values A, B, D and E with + or -, as derived earlier.

A=12*EI/L^3 B=6*EI/L^2 D=4*EI/L E=2*EI/L

member 1.

A=12EI/2,00^3= 12EI/8,00= 1,50 EI

B= 6EI/2,00^2= 6EI/4,00= 1,50 EI

D=4EI/2,00= 2,00 EI

E=2EI/2,00= 1,00 EI

$$\begin{bmatrix} A & B & -A & B \\ B & D & -B & E \\ -A & -B & A & -B \\ B & E & -B & D \end{bmatrix}$$

Member 2.

A=12EI/2,40^2= 12EI/13,82= 0,87 EI

B= 6EI/2,49^2= 6EI/5,76 = 1,04 EI

D= 4EI/2,40= 1,67 EI E=2EI/2,40= 0,84 EI

UV1, UR1, UV3 and UR3 are known, prescribed, all zero. Two equations with unknowns UV2 and UR2 remain.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{237} & \underline{-46} & 0 & 0 \\ 0 & 0 & \underline{-46} & \underline{367} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} UV1 \\ UR1 \\ UV2 \\ UR2 \\ UV3 \\ UR3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

x EI/100 CC $\underline{u}$ $\underline{f}$

The vertical member end forces are assumed downward, so directed on the separated joints directed upward.

Fig.2

Σ vert. joint 2=0 F21+F23-6=0 F21+F23= 6

2,37*UV2 -0,46*UR2 = 6

-0,46*UV2 +3,67*UR2 = 0 from which follow

UV2= 2,59/EI m and UR2= 0,33/EI rad.

Member 2.
Fig.4.
F23= 0,87(2,59) +1,04(0,33)= 2,59 kN
M23= 1,04(2,59) +1,67(0,33)= 3,24 kNm

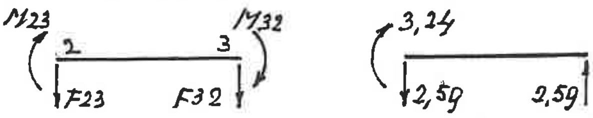


Fig.4.

F32= -0,87(2,59) -1,04(0,33)= -2,59 kN
M32= 1,04(2,59) +0,84(0,33)= 2,97 kNm

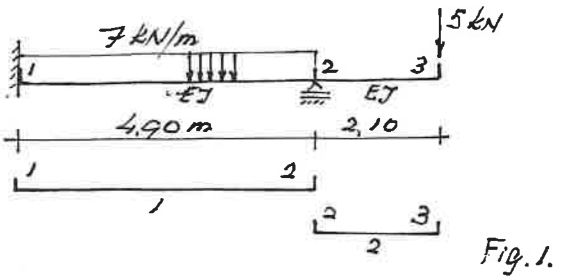


Fig. 1.

$$
\begin{bmatrix} F12\\ M12\\ F21\\ M21 \end{bmatrix} =
\begin{array}{cc}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}\\
\begin{array}{c} 1\\ 2\\ 3\\ 4 \end{array} &
\begin{bmatrix}
102 & 250 & -102 & 250\\
250 & 816 & -250 & 408\\
-102 & -250 & \underline{102} & \underline{-250}\\
250 & 408 & \underline{-250} & \underline{816}
\end{bmatrix}
\end{array}
$$

S51

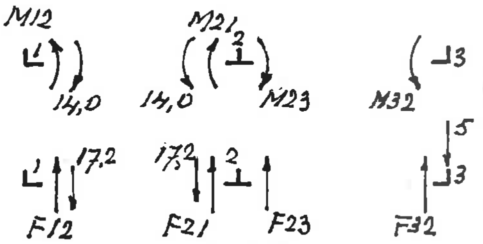


Fig. 2.

$$
\begin{bmatrix} F23\\ M23\\ F32\\ M32 \end{bmatrix} =
\begin{array}{cc}
 & \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array}\\
\begin{array}{c} 3\\ 4\\ 5\\ 6 \end{array} &
\begin{bmatrix}
\underline{1296} & \underline{1361} & -1296 & 1361\\
\underline{1361} & \underline{1905} & -1361 & 952\\
-1296 & -1361 & 1296 & -1361\\
1361 & 952 & -1361 & 1905
\end{bmatrix}
\end{array}
$$

S52

$$
\begin{array}{cc}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}\\
\begin{array}{c} 1\\ 2\\ 3\\ 4\\ 5\\ 6 \end{array} &
\begin{bmatrix}
102 & 250 & -102 & 250 & . & .\\
250 & 816 & -250 & 408 & . & .\\
-102 & -250 & \underline{1398} & \underline{1111} & -1296 & 1361\\
250 & 408 & \underline{1111} & \underline{2721} & -1361 & 952\\
. & . & -1296 & -1361 & 1296 & -1361\\
. & . & 1361 & 953 & -1361 & 1905
\end{bmatrix}
\end{array}
$$

x EI/1000 CC

A check with 4th row of CC, witjout EI and zero multiplications, UV1=0, UR1=0 and UV2=0.

M21+M23=
2,721*UR2 -1,361*UV3 +0,952*UR3 =

2,721(-4,28) -1,361(6,44) +0,952(6,74 =

-11,65 -8,76 +6,39 = 14,02 ≈ 14,0 kNm !

Example.

Fig.1.
For both members applies the same member stiffness matrix S5 on the right, lowest member end number A on the left, highest member end number B on the right, see page 40 .

$$
\begin{bmatrix}
A & B & -A & B\\
B & D & -B & E\\
-A & -B & A & -B\\
B & E & -B & D
\end{bmatrix}
$$

<u>Member 1 with L=4,90 m.</u>
A= 12*EI/L^3= 12*EI/(4,90)^3= 0,102 *EI
B= 6*EI/L^2= 6*EI/(4,90)^2= 0,250 *EI

D= 4*EI/L= 4*EI/4,90= 0,816 *EI
E= 2*EI/L= 2*EI/4,90= 0,408 *EI

<u>Member 2 with L=2,10 m</u>, simalar way.
A= 1,296 *EI B= 1,361 *EI
D= 1,905 *EI E= 0,952 *EI

Elements of force vector <u>f</u> are the joint load forces and joint load moments.
Fig.2.
7 kN/m gives joint moments (page 40) at joint 1 en 2, (1/12)*7*(4,90)^2= 14,0 kNm, and joint forces (1/2)*7*4,90= 17,2 kN.

Σ vert. joint 1=0	F12-17,2=0	F12= 17,2 kN
Σ mom. joint 1=0	M12-14,0=0	M12= 14,0 kNm

Σ vert. joint 2=0
F21+F23-17,2=0 kN F21+F23= 17,2 kN
Σ mom. joint 2=0
M21+M23+14,0=0 M21+M23= -14,0 kNm

Σ vert. joint 3=0	F32-5,0=0	F32= 5,0 kN
Σ mom. joint 3=0		M32= 0 kNm

UV1=0, UR1=0 and UV2=0, the concerning diagonal elements are 1, rows and columns zero, and in the force vector the values of the known displacements.
The unknown 'displacements' are UR2, UV3 and UR3 which can be found with the remaining three equations.

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 2721 & -1361 & 952\\
0 & 0 & 0 & -1361 & 1296 & -1361\\
0 & 0 & 0 & 953 & -1361 & 1905
\end{bmatrix}
\cdot
\begin{bmatrix} UV1\\ UR1\\ UV2\\ \underline{UR2}\\ \underline{UV3}\\ \underline{UR3} \end{bmatrix}
=
\begin{bmatrix} 0\\ 0\\ 0\\ \underline{-14,0}\\ \underline{5,0}\\ \underline{0} \end{bmatrix}
$$

x EI/1000 CC <u>u</u> <u>f</u>

EI(2,721*UR2 -1,361*UV3 +0,952*UR3)= -14,0

EI(-1,361*UR2 +1,296*UV3 -1,361*UR3)= 5,0

EI(0,952*UR2 -1,361*UV3 +1,905*UR3)= 0

With computer Gauss then follow

<u>UR2= -4,28/EI</u> <u>UV3= 6,44/EI</u> <u>UR3= 6,74/EI</u>.

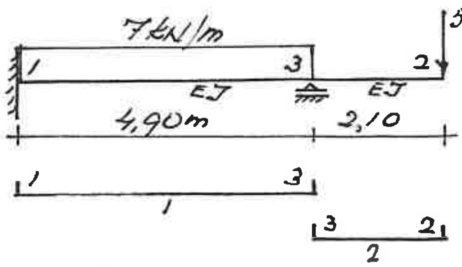


Fig. 3.

$$
\begin{array}{c}
\\
\begin{bmatrix} F13 \\ M13 \\ F31 \\ M31 \end{bmatrix}
\end{array}
=
\begin{array}{cccc}
1 & 2 & 5 & 6 \\
\end{array}
\begin{bmatrix} 102 & 250 & -102 & 250 \\ 250 & 816 & -250 & 408 \\ -102 & -250 & \underline{102} & \underline{-250} \\ 250 & 408 & \underline{-250} & \underline{816} \end{bmatrix}
\cdot
\begin{bmatrix} UV1 \\ UR1 \\ UV3 \\ \underline{UR3} \end{bmatrix}
$$

S51

$$
\begin{bmatrix} F23 \\ M23 \\ F32 \\ M32 \end{bmatrix}
=
\begin{array}{cccc}
3 & 4 & 5 & 6 \\
\end{array}
\begin{bmatrix} 1296 & -1361 & -1296 & -1361 \\ -1361 & 1905 & 1361 & 952 \\ -1296 & 1361 & \underline{1296} & \underline{1361} \\ -1361 & 952 & \underline{1361} & \underline{1905} \end{bmatrix}
\cdot
\begin{bmatrix} \underline{UV2} \\ \underline{UR2} \\ UV3 \\ \underline{UR3} \end{bmatrix}
$$

S52

$$
\begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 102 & 250 & . & . & -102 & 250 \\
2 & 250 & 816 & . & . & -250 & 408 \\
3 & . & . & 1296 & -1361 & -1296 & -1361 \\
4 & . & . & -1361 & 1905 & 1361 & 952 \\
5 & -102 & -250 & -1296 & 1361 & \underline{1398} & \underline{1111} \\
6 & 250 & 408 & -1361 & 952 & \underline{1111} & \underline{2721}
\end{array}
$$

x EI/1000 CC

Elements of $\underline{f}$ <u>before</u> the alteration.

Σ vert. joint 1=0
F13-17,2=0 F13= 17,2 kN

Σ mom. joint 1=0
M13-14,0=0 M13= 14,0 kNm

Σ vert. joint 2=0
F23-5=0 F23= <u>5,0</u> kN

Σ mom. joint 2=0 M32= 0,0 kNm

Σ vert. joint 3=0
F31+F32-17,2=0 F31+F32= <u>17,2</u> kN

Σmom. joint 3=0
M31+M32+14,0=0 M31+M32= <u>-14,0</u> kNm

Fig.3.
The same construction of the preceding page with a different order of the joint numbers, 1-3-2 instead of 1-2-3.

<u>Member 1.</u>

Like on the preceding page with on the left the lowest member end number 1 and the highest member end number 3 on the right. See page 42.

$$
\begin{bmatrix} A & B & -A & B \\ B & D & -B & E \\ -A & -B & A & -B \\ B & E & -B & D \end{bmatrix}
$$

<u>Staaf 2.</u>

Now the lowest member end number 2 on the right, the highest member end number 3 on the left.
See derivation of S5 page 43.

$$
\begin{bmatrix} A & -B & -A & -B \\ -B & D & B & E \\ -A & B & A & B \\ -B & E & B & D \end{bmatrix}
$$

The letter values for both S5's are equal, only the order is different. S52 of member 2 now looks different.
<u>A=12*EI/L^3 B=6*EI/L^2 D=4*EI/L E=2*EI/L</u>

On the left S51 and S52 are put in CC. Since UV1-0, UR1=0 and UV3=0 the three unknown UV2, UR2 and UR3 are left to be calculated.
Here below the three equations are visible.

$$
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1296 & -1361 & 0 & -1361 \\ 0 & 0 & -1361 & 1905 & 0 & 952 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1361 & 952 & 0 & 2721 \end{bmatrix}
\cdot
\begin{bmatrix} UV1 \\ UR1 \\ \underline{UV2} \\ \underline{UR2} \\ UV3 \\ \underline{UR3} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ \underline{5} \\ \underline{0} \\ 0 \\ \underline{-14,0} \end{bmatrix}
$$

x EI/1000 CC $\underline{u}$ $\underline{f}$

The elements of force vector $\underline{f}$ are found with Σ vert.=0 and Σhor. =0 of the joints. They are replaced by the given values of the prescribed knowns.
See on the left the values of the elements of $\underline{f}$ <u>before</u> the alteration of $\underline{f}$ here above.

EI(1,296*UV2 -1,361*UR2 -1,361*UR3)= 5,0

EI(-1,361*UV2 +1,905*UR2 +0,952*UR3)= 0,0

EI(-1,361*UV2 +0,952*UR2 +2,721*UR3= -14,0

With computer Gauss then follow

<u>UV2= 6,44/EI</u> <u>UR2= 6,74/EI</u> <u>UR3= -4,28/EI</u>.

On the preceding pagewas found
<u>UV3= 6,44/EI</u> <u>UR3= 6,74/EI</u> <u>UR2= -4,28/EI</u>

On page 40 the same construction with one member without joint displacement plus a separate calculation of the overhanging member.
<u>UV3= 6,42/EI</u> <u>UR3= 6,74/EI</u> <u>UR2= -4,29/EI</u>

5. Continuous beams over more than two supports with vertical joint displacements and with member ends not regarded as joints but as hinges.

Fig.1.
Beam/member end A regarded as hinge, not as a real joint.

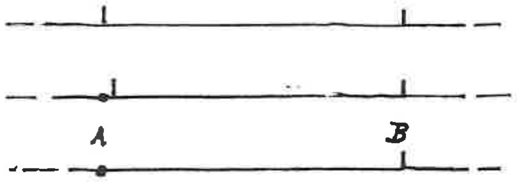
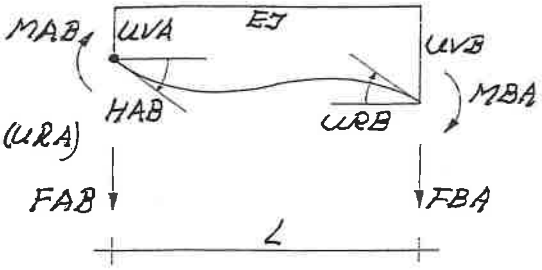


Fig.1.

Fig.2a.
Like figure 2a of page 41 with the same assumptions for UVA and UVB, member end moments MAB and MBA and joint rotation URB.
The slope deflection HAB at A will be separately calculated after the other unknows have become known.



=

Fig.2a.

Fig.2b.
Member end B is hold and A with the assumed displacement UVA displaced downward. With the caused deformation one sees that the moment MB1 is to the right. For equilibrium the drawn forces FA1 and FB1 form a couple of forces to the left. With the formulas on page 97 follow

MB1= (3*EI/L^2)*UVA   and   MA1=0,

and with the equations of equilibrium follow the forces FA1 and FB1

FA1= (3*EI/L^3)*UVA   and   FB1= (3*EI/L^3)*UVA.

At A arises slope deflection HA1 to the left,
HA1= (3/2*L))*UVA.

Now there is not the case of an applied rotation URA like on figure 2c on page 41.
Therefore MA2, MB2, FA2 and FB2 are missing.

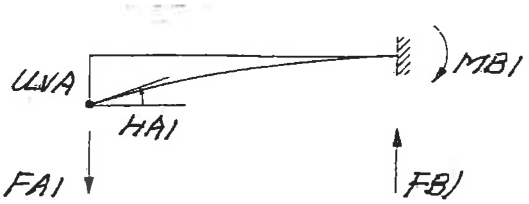


Fig.2b.

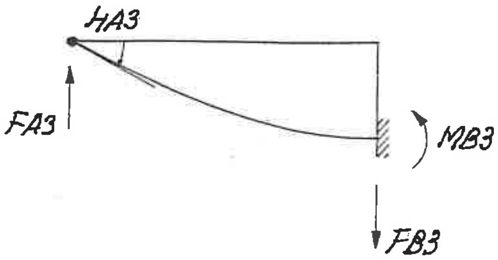
+

Fig.2c.
The member end on the right is hold and displaced over UVB downward. The deformation of the member causes a moment MB3 to the left and with equilibrium then follow the forces FA3 and FB3 forming a couple of forces to the right.

MB3= (3*EI/L^2)*UVB   and   MA3=0

FA3= (3*EI/L^3)*UVB   and   FB3= (3*EI/L^3)*UVB.

At A arises slope deflection HA3 to the right,
HA3= (3/(2*L))*UVA



+

Fig.2c.

Fig.2d.
Now rotation URB is applied as assumed to the right. Then the deformation can arise only if moment MB4 is to the right.
The formula gives URB= (MB4*L)/(3*EI) so that

MB4= (3*EI/L)*URB.

From equilibrium follows FA4 downward and FB4 upward.

FA4= (3*EI/L^2)*URB   and   FB4= (3*EI/L^2)*URB.

At A arises and angle half as large as URB,
HA4= URB/2 to the left.

(...all without sign conventions...)

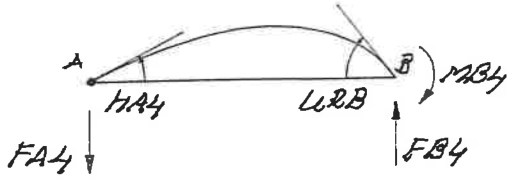


Fig.2d.

$$\begin{bmatrix} FAB \\ MAB \\ FBA \\ MBA \end{bmatrix} = S5 \cdot \begin{bmatrix} UVA \\ URA \\ UVB \\ URB \end{bmatrix}$$

$\underline{f}$ S5 $\underline{u}$

$$\begin{bmatrix} 3EI/L^3 & 0 & -3EI/L^3 & 3EI/L^2 \\ 0 & 0 & 0 & 0 \\ -3EI/L^3 & 0 & 3EI/L^3 & -3EI/L^2 \\ 3EI/L^2 & 0 & -3EI/L^2 & 3EI/L \end{bmatrix}$$

S5

$$\begin{bmatrix} A & 0 & -A & B \\ 0 & 0 & 0 & 0 \\ -A & 0 & A & -B \\ B & 0 & -B & D \end{bmatrix}$$

Fig.3a.

$$\begin{bmatrix} 14 & 0 & 14 & 83 \\ 0 & 0 & 0 & 0 \\ -14 & 0 & 14 & -83 \\ 83 & 0 & -83 & 500 \end{bmatrix}$$

x EI/1000

$$\begin{bmatrix} A & B & -A & B \\ B & D & -B & E \\ -A & -B & A & -B \\ B & E & -B & D \end{bmatrix}$$

$$\begin{bmatrix} 56 & 167 & -56 & 167 \\ 167 & 667 & -167 & 333 \\ -56 & -167 & 56 & -167 \\ 167 & 333 & -167 & 667 \end{bmatrix}$$

Fig.3b.

Next the elements of member stiffness matrix S5 can be determined by addition of forces and moments of the figures as shown on page 41 . Now the second figure is missing because joint rotation URA is missing, therefore
FA2=0, MA2=0, FB2=0, MB2=0 en HA2=0.

<u>FAB= FA1 +FA2 -FA3 +FA4</u>

= (3*EI/L^3)*UVA +0 *URA

-(3*EI/L^3)*UVB +(3*EI/L^2)*URB.

Member end moment on the left MA2=0 because of the hinge, written out then follows the equation

MAB= 0*UVA +0*URA +0*UVB +0*URB.

Next the third equation

FBA= FB2 +FB3 -FB1 -FB4 and in correct order

<u>FBA=-FB1 +FB2 -FB3 +FB4</u>

=-(3*EI/L^3)*UVA +0 *URA

+(3*EI/L^3)*UVB -(3*EI/L^2)*URB and

Finally the fourth equation

<u>MBA= MB1 +MB2 -MB3 +MB4</u>

= (3*EI/L^2)*UVA +0 *URA

-(3*EI/L^2)*UVB +(3*EI)/L)*URB

On the left the elements of S5 are in matrix form given. There are three different values represented with the following capitals

<u>A=3*EI/L^3 B=3*EI/L^2 D=3*EI/L</u>.

The separately slope deflection to be calculated at hinge A is HAB assumed direction to the right. The sum of the angles of the figures is

HAB= HA2 +HA3 -HA1 -HA4 or

<u>HAB=-HA1 +HA2 +HA3 -HA4</u>

= -(3/(2*L))*UVA + 0*URA

+(3/(2*L))*UVB - URB/2 or

<u>HAB= 1,5*(UVB-UVA)/L -URB/2</u>.

Fig.3a.
A as hinge and B as joint.
A=3EI/L^3= 0,014 B=3EI/L^2= 0,083
D=3EI/L= 0,500

Fig.3b.
A and B regarded as joints, then become, see page 42 , the elements of S5 with L=6 m,

A=12EI/L^3= 0,056 B=6EI/L^2= 0,167

D=4EI/L= 0,667 E=2EI/L= 0,333

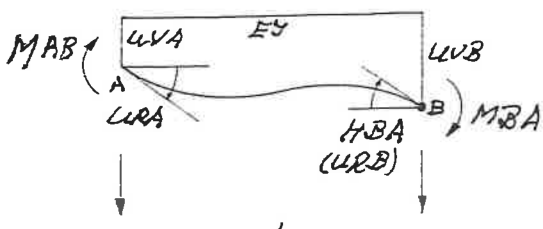


Fig.3a.

=

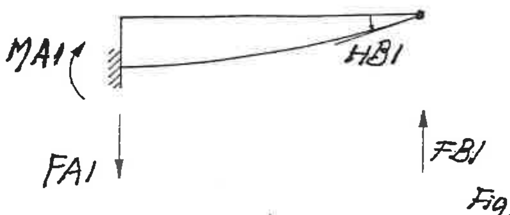


Fig.3b.

+

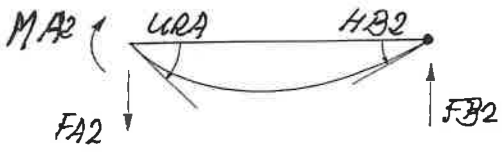


Fig.3c.

+

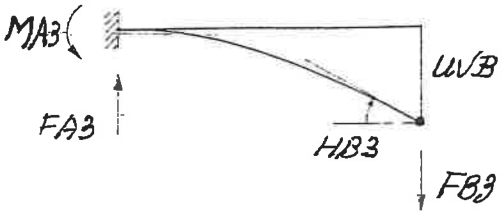


Fig.3d.

$$\begin{bmatrix} 3EI/L\text{^}3 & 3EI/L\text{^}2 & -3EI/L\text{^}3 & 0 \\ 3EI/L\text{^}2 & 3EI/L & -3EI/L\text{^}2 & 0 \\ -3EI/L\text{^}3 & -3EI/L\text{^}2 & 3EI/L\text{^}3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\underline{f}}{\begin{bmatrix} FAB \\ MAB \\ FBA \\ MBA \end{bmatrix}} = \underset{S5}{\begin{bmatrix} A & B & -A & 0 \\ B & D & -B & 0 \\ -A & -B & A & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \cdot \underset{\underline{u}}{\begin{bmatrix} UVA \\ URA \\ UVB \\ URB \end{bmatrix}}$$

Fig.3a.
Next member end B regarded as a hinge and A as a real joint.
Order of the displacements to apply UVA, URA, UVB. Not URB since B is a hinge. Slope deflection HBA is separately calculated.

Fig.3b
Due to displacement UVA arises at A due to the deformation moment MA1 to the right.

MA1= (3*EI/L^2)*UVA.

With the equilibrium equations follow

FA1= (3*EI/L^2)*UVA   en   FB1= (3*EI/L^2)*UVA.

Angle HB1 is HB1=1,5*UVA/L.

Fig.3c.
Moment MA2 belongs to joint rotation URA.

MA2= (3*EI/L)*URA and the forces FA2 and FB2

which make equilibrium with MA2.

FA2= (3*EI/L^2)*URA and FB2= (3*EI/L^2)*URA.

Angle HB2 is HB2=URA/2.

Fig.3d.
By displacement UVB arises by the deformation at A a moment MB3 to the left.

MA3= (3*EI/L^2)*UVB   and the forces

FA3= (3*EI/L^3)*UVB and FB3= (3*EI/L^3)*UVB.

There is no angle URB at B, no fourth case, FA4=0, MA4=0, FB4=0 and MB4=0.

Addition of the figures 4b, 4c en 4d gives FAB, MAB en FBA.

```
FAB= FA1 +FA2 -FA3 +FA4
      (3*EI/L^3)*UVA + (3*EI/L^2)*URA
     -(3*EI/L^3)*UVB        +0    *URB

MAB= MA1 +MA2 -MA3 +MA4
      (3*EI/L^2)*UVA +   (3*EI/L)*URA
     +(3*EI/L^2)*UVB        +0    *URB

FBA=-FB1 -FB2 +FB3 + FB4
    =-(3*EI/L^3)*UVA + (3*EI/L^2)*URA
     +(3*EI/L^3)*UVB        +0    *URB

MBA= 0*UVA + 0*URA + 0*UVB + 0*URB
```

On the left in matrixvorm, and the elements represented with the capitals A, B and D.

A=3*EI/L^3   B=3*EI/L^2   D=3*EI/L

Slope deflection HBA at B assumed to the right,

HBA= -HB1 -HB2 +HB3 +HB4

   = -1,5*UVA/L -URA/2 +1,5*UVB/L + 0*URB

HBA= 1,5*(UVB-UVA)/L -URA/2

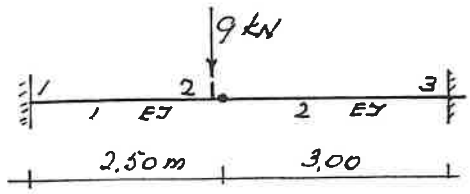
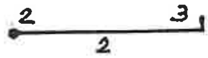


Fig. 1.

$$\begin{bmatrix} A & B & -A & B \\ B & D & -B & E \\ -A & -B & A & -B \\ B & E & -B & D \end{bmatrix} \qquad \begin{bmatrix} A & 0 & -A & B \\ 0 & 0 & 0 & 0 \\ -A & 0 & A & -B \\ B & 0 & -B & D \end{bmatrix}$$

page   page



$$\begin{bmatrix} F12 \\ M12 \\ F21 \\ M21 \end{bmatrix} = \begin{bmatrix} 768 & 960 & -768 & 960 \\ 960 & 1600 & -960 & 800 \\ -768 & -960 & \underline{768} & \underline{-960} \\ 960 & 800 & \underline{-960} & \underline{1600} \end{bmatrix} \cdot \begin{bmatrix} UV1 \\ UR1 \\ \underline{UV2} \\ \underline{UR2} \end{bmatrix}$$

x EI/1000 S51

$$\begin{bmatrix} F23 \\ M23 \\ F32 \\ M32 \end{bmatrix} = \begin{bmatrix} \underline{111} & \underline{0} & -111 & 333 \\ \underline{0} & \underline{0} & 0 & 0 \\ -111 & 0 & 111 & -333 \\ 333 & 0 & -333 & 1000 \end{bmatrix} \cdot \begin{bmatrix} \underline{UV2} \\ \underline{UR2} \\ UV3 \\ UR3 \end{bmatrix}$$

x EI/1000 S52

$$\begin{bmatrix} 768 & 960 & -768 & 960 & . & . \\ 960 & 1600 & -960 & 800 & . & . \\ -768 & -960 & \underline{879} & \underline{-960} & -111 & 333 \\ 960 & 800 & \underline{-960} & \underline{1600} & 0 & 0 \\ . & . & -111 & 0 & 111 & -333 \\ . & . & 333 & 0 & -333 & 1000 \end{bmatrix} \cdot \begin{bmatrix} UV1 \\ UR1 \\ \underline{UV2} \\ \underline{UR2} \\ UV3 \\ UR3 \end{bmatrix}$$

CC

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 879 & -960 & 0 & 0 \\ 0 & 0 & -960 & 1600 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} UV1 \\ UR1 \\ UV2 \\ UR2 \\ UV3 \\ UR3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Slope deflection H23 at member end 2 of member 2, see page ,

<u>H23=</u> 1,5*(UV3-UV2)/3,00 -UR3/2=

= 1,5*(0-29,70/EI)/3,00 -0/2= <u>-14,9/EI</u>.

Example.

Fig.1.
Joint 2 is assumed left of the hinge, short stripe. This is one of three possibilities.

The joint load moments and joint load forces are all zero except joint 2 with a vertical joint load force of 9 kN.

<u>Member 1.</u>

Both member end regarded as joints, the member ends are rigidly connected with the joints. Then the member stiffness matrix on the left S51 of page 44 applies.

A= 12*EI/L^3= 12*EI/(2,5)^3= 0,768 *EI
B= 6*EI/L^2= 6*EI/(2,5)^2= 0,960 *EI

D= 4*EI/L= 4*EI/2,5= 1,600 *EI
E= 2*EI/L= 2*EI/2,5= 0,800 *EI

<u>Member 2.</u>

Member end 3 is a joint and left member end 2 is a hinge. Then stiffness matrix S52 of page 48 shown on the left applies.

A= 3*EI/L^3= 3*EI/(3,0)^3= 0,111 *EI
B= 3*EI/L^2= 3*EI/(3,0)^2= 0,333 *EI
D= 3*EI/L = 3*EI/(3,0) = 1,000 *EI

Both member stiffness matrices S51 and S52 composed give construction stiffness matrix CC.

Since UV1, UR1, UV3 and UR3 are prescrbed and zero, two equations with UV2 and UR2 remain to be solved. If a prescribed 'displacement' not zero then the concerning column of CC alters.

0,879*UV2 -0,960*UR2 = 9

-0,960*UV2 +1,600*UR2 = 0 from which

<u>UV2= 29,7/EI m</u> and <u>UR2= 17,8/EI rad</u>.

<u>Member 1.</u>

F12= EI(-0,768(29,7/EI) +0,960(17,8/EI))=
-22,81 +17,09= -5,72 kN

M12= EI(-0,960(29,7/EI) +0,800(17,8/EI))=
-28,54 +14,24= -14,30 kNm

F21= EI(0,786(29,7/EI) -0,960(17,8/EI))=
= 22,81 -17,09= 5,72 kN

M21= EI(-0,960(29,7/EI) +1,600)17,8/EI))=
= -28,51 +28,48= -0,03 is 0.

<u>Member 2.</u>

F23= 0,111(29,7)= 3,30 kN   M23=0 kNm

F32= -0,111(29,7)= -3,30 kN

M32= 0,333(29,7)= 9,89 kNm

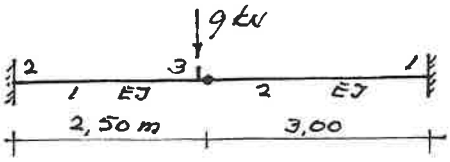
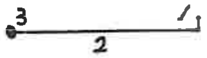


Fig. 1.

$$\begin{bmatrix} A & B & -A & B \\ B & D & -B & E \\ -A & -B & A & -B \\ B & E & -B & D \end{bmatrix} \qquad \begin{bmatrix} A & -B & -A & 0 \\ -B & D & B & 0 \\ -A & B & A & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

page   page



$$\begin{bmatrix} F23 \\ M23 \\ F32 \\ M32 \end{bmatrix} \begin{array}{c} 3 \\ 4 \\ = 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 768 & 960 & -768 & 960 \\ 960 & 1600 & -960 & 800 \\ -768 & -960 & \underline{768} & \underline{-960} \\ 960 & 800 & \underline{-960} & \underline{1600} \end{bmatrix} \end{array} \cdot \begin{bmatrix} UV2 \\ UR2 \\ \underline{UV3} \\ \underline{UR3} \end{bmatrix}$$

x EI/1000   S51

$$\begin{bmatrix} F13 \\ M13 \\ F31 \\ M31 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ = 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} 111 & -333 & -111 & 0 \\ -333 & 1000 & 333 & 0 \\ -111 & 333 & \underline{111} & \underline{0} \\ 0 & 0 & \underline{0} & \underline{0} \end{bmatrix} \end{array} \cdot \begin{bmatrix} UV1 \\ UR1 \\ \underline{UV3} \\ \underline{UR3} \end{bmatrix}$$

x EI/1000   S52

$$\begin{bmatrix} 111 & -333 & . & . & -111 & 0 \\ -333 & 1000 & . & . & 333 & 0 \\ . & . & 768 & 960 & -768 & 960 \\ . & . & 960 & 1600 & -960 & 800 \\ -111 & 333 & -768 & -960 & \underline{879} & \underline{-960} \\ 0 & 0 & 960 & 800 & \underline{-960} & \underline{1600} \end{bmatrix} \cdot \begin{bmatrix} UV1 \\ UR1 \\ UV2 \\ UR2 \\ \underline{UV3} \\ \underline{UR3} \end{bmatrix}$$

CC

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 879 & -960 \\ 0 & 0 & 0 & 0 & -960 & 1600 \end{bmatrix} \cdot \begin{bmatrix} UV1 \\ UR1 \\ UV2 \\ UR2 \\ \underline{UV3} \\ \underline{UR3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Slope deflection H31 at member end 2 of member 2, see page 48 ,
<u>H31=</u> 1,5*(UV1-UV3)/3,00 -UR1/2=

= 1,5*(0-29,70/EI)/3,00 -0/2= <u>-14,9/EI</u>.

<u>Example preceding page with number erder 2-3-1 instead of 1-2-3.</u>

Fig.1.
Joint 2 assumed left of the hinge. A vertical joint load force of 9 kN.

<u>Member 1.</u>

The same stiffness matrix as on the preceding page. Both member end regarded as real joints with F23, M23, F32, M32, en UV2, UR2, UV3 and UR3. Note the member end numbering.

<u>Member 2.</u>

Again the hinge on left end but different member end numbering, with matrix S5 of page with the same A, B and D.

A= 3*EI/L^3= 3*EI/(3,0)^3= 0,111 *EI
B= 3*EI/L^2= 3*EI/(3,0)^2= 0,333 *EI
D= 3*EI/L   = 3*EI/(3,0)   = 1,000 *EI

Matrix S52 of member 2 with hinge, with F13, M13, F31, M31, and UV1, UR1, UV3 and UR3. S51 and S52 composed give construction stiffness matrix CC. For S51 and S52 row and column numbers are written to show where in CC the elements 'appear'.

Since UV1, UR1, UV3 and UR3 are prescribed, they are zero, two equations remain to be solved.

0,879*UV3 -0,960*UR3 = 9

-0,960*UV3 +1,600*UR3 = 0   from which

<u>UV3= 29,7/EI m</u>   and   <u>UR3= 17,8/EI rad</u>.

The member end forces and moments are

F23= -5,72 kN	M23= -14,30 kNm
F32= 5,72 kN	M32= 0 kNm
F13= -3,30 kN	M13= 9,89 kNm
F31= 3,30 kN	M31= 0 kNm

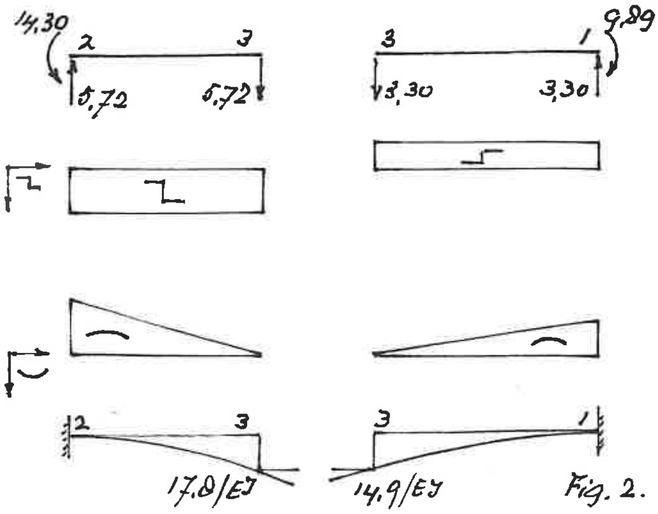


Fig. 2.

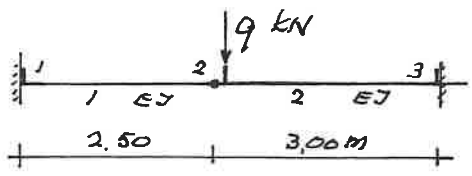
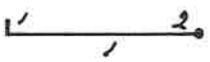
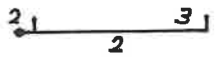


Fig.1.

$$\begin{bmatrix} A & B & -A & 0 \\ B & D & -B & 0 \\ -A & -B & A & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} A & B & -A & B \\ B & D & -B & E \\ -A & -B & A & -B \\ B & E & -B & D \end{bmatrix}$$

page   page





$$\begin{array}{c} \\ \begin{bmatrix} F12 \\ M12 \\ F21 \\ M21 \end{bmatrix} \end{array} \begin{array}{c} \\ \begin{matrix} 1 \\ 2 \\ = \\ 3 \\ 4 \end{matrix} \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 192 & 480 & -192 & 0 \\ 480 & 1200 & -480 & 0 \\ -192 & -480 & \underline{192} & \underline{0} \\ 0 & 0 & \underline{0} & \underline{0} \end{bmatrix} \end{array} \cdot \begin{array}{c} \\ \begin{bmatrix} UV1 \\ UR1 \\ \underline{UV2} \\ \underline{UR2} \end{bmatrix} \end{array}$$

x EI/1000   S51

$$\begin{array}{c} \\ \begin{bmatrix} F23 \\ M23 \\ F32 \\ M32 \end{bmatrix} \end{array} \begin{array}{c} \\ \begin{matrix} 3 \\ 4 \\ = \\ 5 \\ 6 \end{matrix} \end{array} \begin{array}{c} \begin{matrix} 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} \underline{444} & \underline{667} & -444 & 667 \\ \underline{667} & \underline{1333} & -667 & 667 \\ -444 & -667 & 444 & -667 \\ 667 & 667 & -667 & 1333 \end{bmatrix} \end{array} \cdot \begin{array}{c} \\ \begin{bmatrix} \underline{UV2} \\ \underline{UR2} \\ UV3 \\ UR3 \end{bmatrix} \end{array}$$

x EI/1000   S52

$$\begin{bmatrix} 192 & -480 & -192 & 0 & . & . \\ -480 & 1200 & 480 & 0 & . & . \\ -192 & 480 & \underline{636} & \underline{667} & -444 & 667 \\ 0 & 0 & \underline{667} & \underline{1333} & -667 & 667 \\ . & . & -444 & -667 & 444 & -667 \\ . & . & 667 & 667 & -667 & 1333 \end{bmatrix} \cdot \begin{bmatrix} UV1 \\ UR1 \\ \underline{UV2} \\ \underline{UR2} \\ UV3 \\ UR3 \end{bmatrix}$$

CC

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 636 & 667 & 0 \\ 0 & 0 & 0 & 667 & 1333 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} UV1 \\ UR1 \\ \underline{UV2} \\ \underline{UR2} \\ UV3 \\ UR3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Slope deflection H21 at member end 2 of member 1, see page 48 ,
<u>H21=</u> 1,5*(UV2-UV1)/2,5 -UR1/2=

= 1,5*(29,8-0)/2,5 -0/2= 17,9/<u>EI</u>

Example.

Fig.1.
This time joint 2 is assumed on the right instead of on the left of the hinge, see page ,
with the joint load for of 9 kN.

<u>Member 1</u>.
With the member stiffness matrix of page .

A= 3*EI/L^3= 3*EI/(2,5)^3= 0,192 *EI
B= 3*EI/L^2= 3*EI/(2,5)^2= 0,480 *EI
D= 3*EI/L = 3*EI/(2,5) = 1,200 *EI

<u>Member 2</u>.
With the member stiffness matrix of page .

A= 12*EI/L^3= 12*EI/(3,0)^3= 0,444 *EI
B= 6*EI/L^2= 6*EI/(3,0)^2= 0,667 *EI

D= 4*EI/L= 4*EI/3,0= 1,333 *EI
E= 2*EI/L= 2*EI/3,0= 0,667 *EI

The prescribed 'displacements' UV1, UR1, UV3 and UR3, all zero, make two equation remain to solve, EI omitted for convenience.
0,636*UV2 +0,667*UR2= 9

0,667*UV2 +1,333*UR2= 0   from which

<u>UV2= 29,8/EI</u>   en   <u>UR2= -14,9/EI</u>.

Next the member end forces and member end moments are calculated.

Member 1.

F12= -0,192(29,8)= -5,72 kN
M12= 0,480(29,8)= 1430 kNm

F21= 0,192(29,8)= 5,72 kN
M21= 0 kNm

Member 2.

F23= 0,444(29,8) +0,667(-14,9)
   = 13,23 -9,94= 3,29 kN
M23= 0,667(29,8) +1,333(-14,9)
   = 19,88 -19,86= 0,02 is 0 kNm

F32=-0,444(29,8) -0,667(-14,9)
   =-13,23 +9,94= -3,29 kN
M32= 0,667(29,8) +0,667(-14,9)
   = 19,88 -9,94= 9,94 kNm

The reaction forces and moments are the member end forces and moments here below drawn with their real directions.

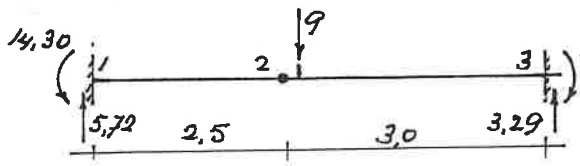


Fig.2.

Σ vert.=0
5,72 -9,00 +3,29= 0,01 is 0.
Σ mom. A=0.
9*2,5 -3,29*5,5 +9,49 -14,30= 0,04 is 0
(Rem. some roundings on the way ...)

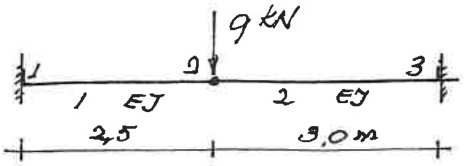
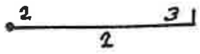


Fig.1.

$$\begin{bmatrix} A & B & -A & 0\\ B & D & -B & 0\\ -A & -B & A & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \qquad \begin{bmatrix} A & 0 & -A & B\\ 0 & 0 & 0 & 0\\ -A & 0 & A & -B\\ B & 0 & -B & D\end{bmatrix}$$

page page



$$\begin{bmatrix}F12\\M12\\F21\\M21\end{bmatrix}\begin{matrix}1\\2\\3\\4\end{matrix} = \begin{array}{cccc} 1 & 2 & 3 & 4\\ \end{array}$$
$$\begin{bmatrix} 192 & 480 & -192 & 0\\ 480 & 1200 & -480 & 0\\ -192 & -480 & \underline{192} & \underline{0}\\ 0 & 0 & \underline{0} & \underline{0}\end{bmatrix}\cdot\begin{bmatrix}UV1\\UR1\\\underline{UV2}\\UR2\end{bmatrix}$$

x EI/1000 S51

$$\begin{bmatrix}F23\\M23\\F32\\M32\end{bmatrix}\begin{matrix}3\\4\\5\\6\end{matrix} = \begin{array}{cccc} 3 & 4 & 5 & 6\\ \end{array}$$
$$\begin{bmatrix} \underline{111} & \underline{0} & -111 & 333\\ \underline{0} & \underline{0} & \underline{0} & \underline{0}\\ -111 & 0 & 111 & -333\\ 333 & 0 & -333 & 1000\end{bmatrix}\cdot\begin{bmatrix}\underline{UV2}\\UR2\\UV3\\UR3\end{bmatrix}$$

x EI/1000 S52

$$\begin{bmatrix} 192 & -480 & -192 & 0 & . & .\\ -480 & 1200 & 480 & 0 & . & .\\ -192 & 480 & \underline{303} & \underline{0} & -111 & 333\\ 0 & 0 & \underline{0} & \underline{0} & 0 & 0\\ . & . & 111 & 0 & 111 & -333\\ . & . & 333 & 0 & -333 & 1000\end{bmatrix}\cdot\begin{bmatrix}UV1\\UR1\\\underline{UV2}\\UR2\\UV3\\UR3\end{bmatrix}$$

CC

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & \underline{303} & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}UV1\\UR1\\\underline{UV2}\\UR2\\UV3\\UR3\end{bmatrix} = \begin{bmatrix}0\\0\\9\\0\\0\\0\end{bmatrix}$$

There is no UR2 because of the hinge, then two slope deflections, H21 and H23, will be calculated separately.

Example.

Fig.1.
This time the hinge itself is vertically loaded with kN. A hinge is not a real joint, there is no joint rotation UR2.

Member 1 with L=2,5 m.

A= 3*EI/L^3= 3*EI/(2,5)^3= 0,192 *EI
B= 3*EI/L^2= 3*EI/(2,5)^2= 0,480 *EI
D= 3*EI/L = 3*EI/(2,5) = 1,200 *EI

Member 2 with L=3,0 m.

A= 3*EI/L^3= 3*EI/(3,0)^3= 0,111 *EI
B= 3*EI/L^2= 3*EI/(3,0)^2= 0,333 *EI
D= 3*EI/L = 3*EI/(3,0) = 1,000 *EI

With prescribed displacements UV1=0, UR1=0, UV3=0 and UR3=0 the construction matrix is altered, four equations of no use.
By coinciding elements of both member matrices, with many zeros, the fourth equation can be missed as well, UR2 falls out. Thus only one equation is left.

0,303*UV2= 9 from which UV2= 29,7/EI.

Member 1 with slope deflection at member end 2,

H21= 1,5*(UV2-UV1)/L -UR1/2

= 1,5*(29,8/EI -0)/2,5 = 17,9/EI.

Member 2 with slope deflection at member end 2,

H23= 1,5*(UV3-UV2)/L -UR3/2

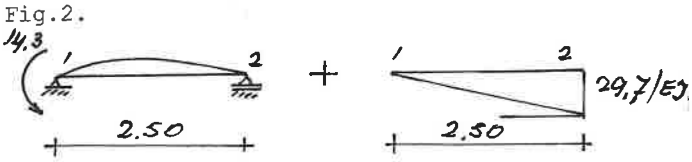
= 1,5*(0- 29,8/EI)/3,0 = -14,9/EI.

Member 1.

Calculation of the member end forces and moments F12, M12, F21 and M21 with help of S51.
F12= -0,129EI*29,7/EI= -5,70 kN

EI omitted , M12= 0,480(29,7)= 14,3 kNm
F21= 5,70 kN M21= 0(29,7)= 0 kNm

Fig.2.



H12= 14,3*2,50/3EI -(29,7/EI)/2,50=
11,9/EI - 11,9/EI = 0,0

H21= 14,3*2,50/6EI +(29,7/EI)/2,50=
6,0/EI + 11,9/EI = 17,9/EI

Applying the 'forget-me-nots'.

UV2= 5,70*2,50^3/(3EI)= 29,7/EI

H21= 5,70*2,50^2/(2EI)= 17,8/EI

Or H21= 1,5(UV2-UV1)/L-UR1/2, see page

H21= 1,5(29,7/EI-0)/2,50-0/2= 17,8/EI.

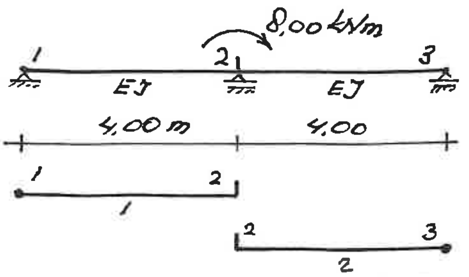


Fig. 1.

$$\begin{bmatrix} F12 \\ M12 \\ F21 \\ M21 \end{bmatrix} = \begin{bmatrix} 47 & 0 & -47 & 188 \\ 0 & 0 & 0 & 0 \\ -47 & 0 & \underline{47} & \underline{-188} \\ 188 & 0 & \underline{-188} & \underline{750} \end{bmatrix} \cdot \begin{bmatrix} UV1 \\ UR1 \\ UV2 \\ \underline{UR2} \end{bmatrix}$$

$$\begin{bmatrix} F23 \\ M23 \\ F32 \\ M32 \end{bmatrix} = \begin{bmatrix} \underline{47} & \underline{188} & -47 & 0 \\ \underline{188} & \underline{750} & -188 & 0 \\ -47 & -188 & 47 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} UV2 \\ \underline{UR2} \\ UV3 \\ UR3 \end{bmatrix}$$

xEI/1000

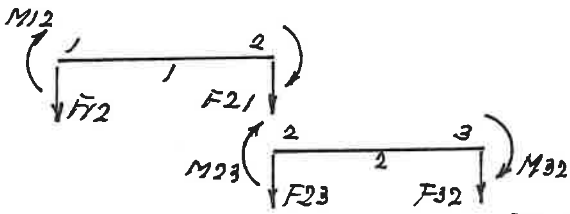


Fig.3.

Member 1.
F12= EI(0,188(5,33/EI))= 1,00 kN
M12= EI(0(5,33/EI))= 0,00 kNm

F21= EI(-0,188(5,33/EI))=-1,00 kN
M21= EI(0,750(5,33/EI))= 4,00 kNm

Member 2.
F23= EI(0,188(5,33/EI))= 1,00 kN
M23= EI(0,750(5,33/EI))= 0,00 kNm

F32= EI(-0,188(5,33/EI))=-1,00 kN
M32= EI(0(5,33/EI))= 4,00 kNm

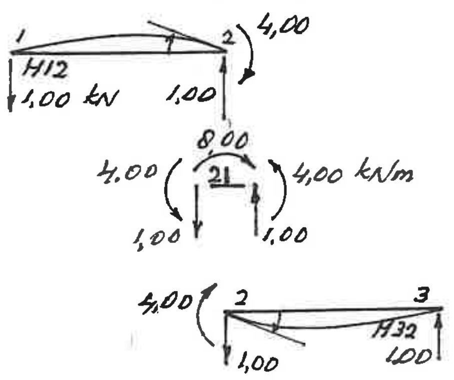


Fig.4.

Example.

Fig.1.
Member ends 1 and 3 regarded as hinges so that joint rotation UR2 is the only unknown.
Slope deflections, or member end rotations, are separately calculated.

Member 1.	Member 2.	
$\begin{bmatrix} A & 0 & -A & B \\ 0 & 0 & 0 & 0 \\ -A & 0 & A & -B \\ B & 0 & -B & D \end{bmatrix}$	$\begin{bmatrix} A & B & -A & 0 \\ B & D & -B & 0 \\ -A & -B & A & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	A=0,047 EI B=0,188 EI D=0,750 EI

L=4,00 m   A=3*EI/L^3   B=3*EI/L^2   D=3*EI/L

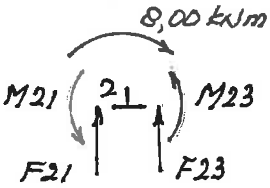


Fig.2.

Joint 2.
Σ mom.=0   M21+M23-8,00=0   M21+M23= 8,00 kNm

Further no joint load moments and joint load forces, also not due to member loads.
And UV1=0, UV2=0 and UV3=0.

$$\begin{bmatrix} 47 & 0 & -47 & 188 & . & . \\ 0 & 0 & 0 & 0 & . & . \\ -47 & 0 & \underline{94} & \underline{0} & -47 & 0 \\ 188 & 0 & \underline{0} & \underline{1500} & -188 & 0 \\ . & . & -47 & -188 & 47 & 0 \\ . & . & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 8,00 \\ 0 \\ 0 \end{bmatrix}$$

x EI/1000   CC $\underline{f}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1500 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} UV1 \\ UR1 \\ UV2 \\ \underline{UR2} \\ UV3 \\ UR3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \underline{8,00} \\ 0 \\ 0 \end{bmatrix}$$

EI(1,500*UR2)= 8,00   <u>UR2= 5,33/EI</u>

Fig.4.
See page 97 for formulas.
H12=4,00*4,00/**6**EI= 2,67/EI to the left, and
H32= 2,67/EI to the left.

For joint rotation UR2, with H21 and H23 follow H21=4,00*4,00/**3**EI= 5,33/EI to the right and H23=4,00*4,00/**3**EI= 5,33/EI to the right like was found, UR2= 5,33/EI.

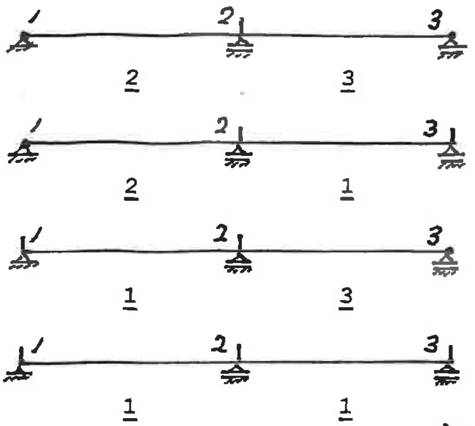


Fig. 5.

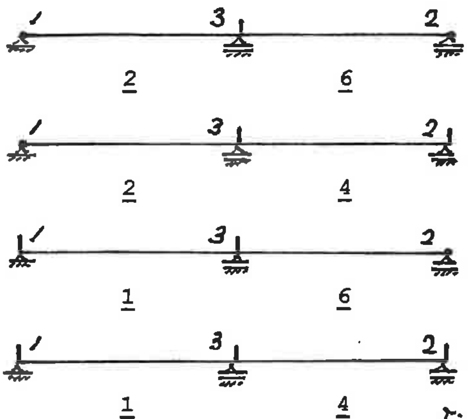


Fig.6.

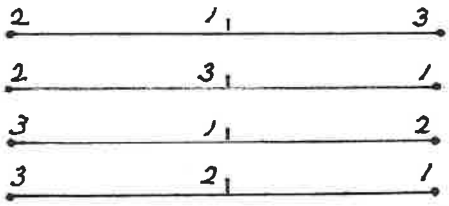


Fig.7.

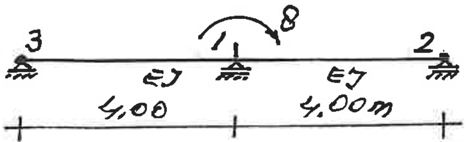


Fig. 8.

$$
\begin{bmatrix}
\underline{94} & \underline{0} & -47 & 0 & -47 & 0 \\
\underline{0} & \underline{1500} & -188 & 0 & 188 & 0 \\
-47 & -188 & 47 & 0 & . & . \\
0 & 0 & 0 & 0 & . & . \\
-47 & 188 & . & . & 0 & 0 \\
0 & 0 & . & . & 0 & 0
\end{bmatrix}
\cdot
\begin{bmatrix}
UV1 \\ \underline{UR1} \\ UV2 \\ UR2 \\ UV3 \\ UR3
\end{bmatrix}
$$

x EI/1000 CC

Joints and hinges, and joint numbering.

Fig.5.
The example of the preceding page has four possible cases with the joint numbering 1-2-3, with assumptions for the member end 1 and 3, joint or hinge.
The added underlined numbers at the concerning member stiffness matrices belong to the members here below. See the six possible member stiffness matrices.

Six possible member stiffness matrices S5!

Six possible S5's depending depending on the joint and member numbering. And the place of the lowest member end number L and the highest member end number H.

$$
\left[\begin{array}{cc|cc}
A & B & -A & B \\
B & D & -B & E \\ \hline
-A & -B & A & -B \\
B & E & -B & D
\end{array}\right]
\quad
\left[\begin{array}{cc|cc}
A & . & -A & B \\
. & . & . & . \\ \hline
-A & . & A & -B \\
B & . & -B & D
\end{array}\right]
\quad
\left[\begin{array}{cc|cc}
A & B & -A & . \\
B & D & -B & . \\ \hline
-A & -B & A & . \\
. & . & . & .
\end{array}\right]
$$

$\underline{1}$ $\underline{2}$ $\underline{3}$

De 2 x 2 deelmatrices van de S5's hieronder met verwisselde L en H zijn gespiegelde deelmatrices van de S5's hierboven t.o.v. de diagonalen.

$$
\left[\begin{array}{cc|cc}
A & -B & -A & -B \\
-B & D & B & E \\ \hline
-A & B & A & B \\
-B & E & B & D
\end{array}\right]
\quad
\left[\begin{array}{cc|cc}
A & -B & -A & . \\
-B & D & B & . \\ \hline
-A & B & A & . \\
. & . & . & .
\end{array}\right]
\quad
\left[\begin{array}{cc|cc}
A & . & -A & -B \\
. & . & . & . \\ \hline
-A & . & A & B \\
-B & . & B & D
\end{array}\right]
$$

$\underline{4}$ $\underline{5}$ $\underline{6}$

$\underline{1}$ and $\underline{4}$ with
A=12EI/L^3 B=6EI/L^2 D=4EI/L E=2EI/L

$\underline{2}$, $\underline{3}$, $\underline{5}$ and $\underline{6}$ with
A=3*EI/L^3 B=3EI/L^2 D=3EI/L

Fig.6.
Four possibilities with numbering 1-3-2.

Fig.7
With three (succesive) numbers there are six arbitrary number combinations. Here are the four remaing combinations given. Each of the four has four possible joint/hinge combinations like figure 5 and 6. All together for this construction 6 x 4 = 24 possible cases all giving the same results. Free to choose.

Fig.8.
One of the possibilities. Just one unknown UR1, also unknown H31 and H21 separately to be calculated. Preceding page with
A=0,047 EI, B=0,188 EI, and D=0,750 EI.
See the concerning matrices $\underline{5}$ with 3-1 H-L, and $\underline{3}$ with 1-2 L-H. On the left construction matrix CC with one equation remaining.
1,500EI*UR1= 8,00 so that UR1= 5,33/EI, like UR2= 5,33/EI of the preceding page.

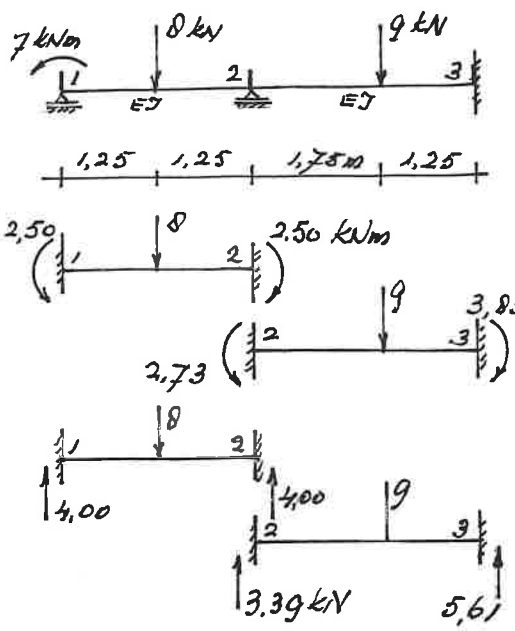


Fig.1.

$$\begin{bmatrix} F12 \\ M12 \\ F21 \\ M21 \end{bmatrix} = \begin{bmatrix} 768 & 960 & -768 & 960 \\ 960 & 1600 & -960 & 800 \\ -768 & -960 & \underline{768} & \underline{-960} \\ 960 & 800 & \underline{-960} & \underline{1600} \end{bmatrix} \cdot \begin{bmatrix} UV1 \\ \underline{UR1} \\ UV2 \\ \underline{UR2} \end{bmatrix}$$

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$$\begin{bmatrix} F23 \\ M23 \\ F32 \\ M32 \end{bmatrix} = \begin{bmatrix} 444 & 667 & -444 & 667 \\ 667 & 1333 & -667 & 667 \\ -444 & -667 & 444 & -667 \\ 667 & 667 & -667 & 1333 \end{bmatrix} \cdot \begin{bmatrix} UV2 \\ \underline{UR2} \\ UV3 \\ UR3 \end{bmatrix}$$

xEI/1000 S52

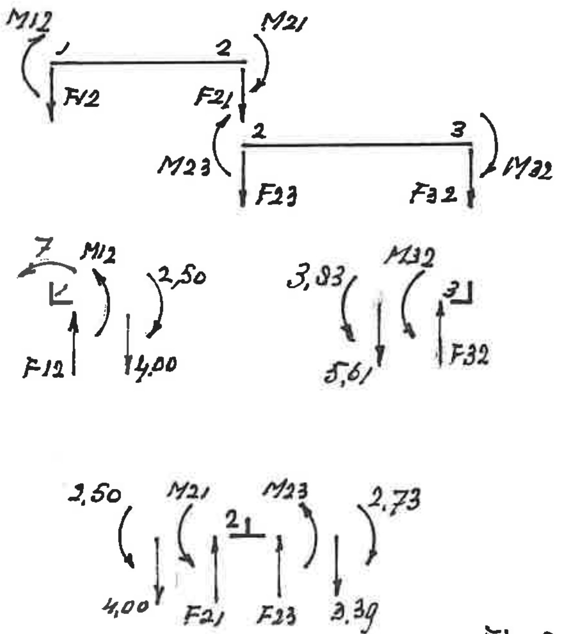

Fig.2.

Example.

Fig.1.
Member end 1 of member 1 cannot be regarded as a hinge because of the joint load moment of 7 kNm, so it is regarded as a real joint.

Member 1.

A=12EI/2,50^3=0,768 EI
B= 6EI/2,50^2=0,960 EI
D=4EI/2,50=1,600 EI
E=2EI/2,50=0,800 EI

$$\begin{bmatrix} A & B & -A & B \\ B & D & -B & E \\ -A & -B & A & -B \\ B & E & -B & D \end{bmatrix}$$

MP12=(1/8)*8*2,50= 2,50 kNm MP21= 2,50 kNm

Member 2.

A=12EI/3,00^3=0,444 EI D=4EI/3,00=1,333 EI
B= 6EI/3,00^2=0,667 EI E=2EI/3,00=0,667 EI

MP23=(9*1,75*1,25^2)/(3,00^2)= 2,73 kNm
MP32=(9*1,25*1,75^2)/(3,00^2)= 3,83 kNm

Determination of the elements of f.
Fig.2.

Joint 1.	Σ vert.=0	F12-4,00=0	F12= 4,00 kN
	Σ mom.=0	M12+7,00-2,50=0	M12=-4,50 kNm
Joint 2.			
	Σ vert.0	F21+F23-4,00-3,39=0	F21+F23= 7,39 kN
	Σ mom.=0	M21+M23+2,50-2,73=0	M21+M23=-0,23 kNm
Joint 3.	Σ vert=0.	F32-5,61=0	F32= 5,61 kN
	Σ mom.=0.	M32+3,83=0	M32=-3,83 kNm

$$\begin{bmatrix} 768 & 960 & -768 & 960 & . & . \\ 960 & 1600 & -960 & 800 & . & . \\ -768 & -960 & \underline{1212} & \underline{-293} & -444 & 667 \\ 960 & 800 & \underline{-293} & \underline{2933} & -667 & 667 \\ . & . & -444 & -667 & 444 & -667 \\ . & . & 667 & 667 & -667 & 1333 \end{bmatrix} \begin{bmatrix} 4,00 \\ \underline{-4,50} \\ 7,39 \\ \underline{0,23} \\ 5,61 \\ -3,83 \end{bmatrix}$$

x EI/1000 CC f

The values of the given displacements UV1=0, UV2=0, UV3=0 and UR3=0 are put/appear in f so that with computer Gauss six equations will be solved.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1600 & 0 & 800 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 800 & 0 & 2933 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} UV1 \\ \underline{UR1} \\ UV2 \\ \underline{UR2} \\ UV3 \\ UR3 \end{bmatrix} \begin{bmatrix} 0 \\ \underline{-4,50} \\ 0 \\ \underline{0,23} \\ 0 \\ 0 \end{bmatrix}$$

EI(1,600*UR1 +0,800*UR2)= -4,50
EI(0,800*UR1 +2,933*UR2)= 0,23 from which

UR1= -3,30/EI and UR2=0,98/EI.